A note on fuzzy preference structures

Dana Hliněná

Dept. of Mathematics, FEEC, Brno University of Technology

Technická 8, 616 00 Brno, Czech Rep.

E-mail: hlinena@feec.vutbr.cz

Abstract

In this paper we present a problem of multicriterial optimization and different models to solve it. One practical example with a solution will be given.
Keywords: Multicriteria optimization, Preference structures, Fuzzy preference structures, Fuzzy logic programming.

1 Introduction to preference structures and fuzzy preference structures

The preference structure is a basic step of preference modeling. Given two alternatives, decision maker defines three binary relation-preference, indifference and incomparability.

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS) a decision-maker makes three decission for any par (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decision define a triplet P, I, J of a crisp binary relations on $\boldsymbol{A}$:

1) a is prefered to $b \Leftrightarrow (a, b) \in P$ (strict preference).

2) a and b are indifferent $\Leftrightarrow (a, b) \in I$ (indifference).

3) a and b are incomparable $\Leftrightarrow (a, b) \in J$ (incomparability).

A preference structure (PS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, P and J are irreflexive.

(ps2) P is asymmetric, I and J are symmetric.

(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$.

(ps4) $P \cup I \cup J \cup P^t = A \times A$ where $P^t(x, y) = P(y, x)$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the large preference relation. The relation R can be interpreted as

$$(a,b) \in R \Leftrightarrow a \text{ is prefered to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J$$

where $coR(a,b) = 1 - R(a,b)$ and

$$P = R \cap co(R^t), I = R \cap R^t, J = co(R) \cap co(R^t).$$

It allows us to construct a preference structure (P, I, J) from a reflexive binary operation R only.

Decision-makers are often uncertain even inconsistent in their judgements. The restriction on two-valued relations have been an important drawback to their practical use. A natural demand led researchers to the introduction of a fuzzy preference structure (FPS). The original idea of using numbers between zero and one to describe the strenth of links between two alternatives goes back to Menger. The introduction of fuzzy relations allowed to express degrees of preference, indifference and incomparability. Of course, the attempts simply to replace the notion used in the definition of (PS) by their fuzzy equivalents have met some problems.

To define (FPS) it is necessary to consider some fuzzy connectives. We shall consider a continuous De Morgan triple (T, S, N) consisting of a continuous t-norm T, continuous t-conorm S and a strong negator N such that $T(x,y) = N(S(N(x), N(y)))$. The main problem consists in the fact that the completeness condition (ps4) can be written in many forms, e.g.:

$$co(P \cup P^t) = I \cup J, P = co(P^t \cup I \cup J), P \cup I = co(P^t \cup J).$$

Let (T,S,N) be De Morgan triplet. A fuzzy preference structure (FPS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary fuzzy relations on $\boldsymbol{A}$ such that

(f1) I is reflexive, P and J are irreflexive. $I(a,a) = 1, P(a,a) = J(a,a) = 0$

(f2) P is T-asymmetrical, I and J are symmetrical. $T(P(a,b), P(b,a)) = 0$

(f3) $T(P,I) = T(P,J) = T(I,J) = 0.$ for all pair of alternatives

(f4) $(\forall(a,b) \in A^2) S(P, P^t, I, J) = 1$ or $N(S(P,I)) = S(P^t, J)$ or another completeness conditions.

Our contribution is organized as follows: In Section 2 we introduce an example taken from [2]. Then in Section 3 we give our new solution to the demonstration example.

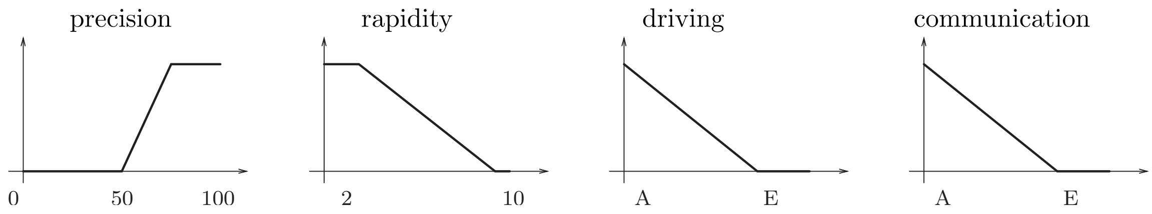
2 Motivation Example

In this paper we illustrate (on an example taken from Michel Grabisch, Marc Roubens [2]) several aspects of a multicriterial problem that we try to approach from different perspectives – deductive, inductive, and different formal models – Choquet integrals, fuzzy preference structures.

Table 1: Performances of the different trainees

name	*precision (%)*	*rapidity (tu)*	*driving*	*communication*
Arthur	90	2	B	D
Lancelot	80	4	B	B
Yvain	95	5	C	A
Perceval	60	6	B	B
Erec	65	2	C	B

Table 2: Scores on the different criteria



In [2] the authors consider the problem of the evaluation of trainees learning to drive military vehicles. The instructors evalueted the trainees according to 4 criteria:

Firing precision: The percentage of success during the exercise is computed.

Target detection rapidity: The elapsed time between the appearance of the target and the detection is measured in tu (time unit).

Driving: In order to go from one point to another, the trainee has to choose a suitable trajectory, or to follow a given one as precisely as possible. A qualitive score is given by the instructor, ranging from A (excelent) to E (hopeless).

Communication: The trainee is supposed to belong to some unit, and thus he should understand and obey orders, and also report actions. As for the driving criterion, a qualitative score is given by the instructor, ranging from A (perfect) to E (incontrollable).

In this example, Grabisch and Roubens consider 5 trainees, whose names and performances on each criterion are given in Table 1.

Instructor's comments:

C.1 (precision): over 90% of success is perfect, below 50% is totally unacceptable.

C.2 (rapidity): below 2 tu is perfect, over 10 tu is totally unacceptable.

C.3 and C.4: these criteria are already expressed in the form of a numerical score.

This permits us to draw utility functions which give the following numerical scores for the trainees in Tables 2, 3.

Table 3: Numerical scores on criteria

name	*precision*	*rapidity*	*driving*	*communication*
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750

Table 4: Ranking of the five trainees

name	*class*	*rank in the class*
Arthur	bad	2
Lancelot	good	1
Yvain	good	2
Perceval	bad	1
Erec	average	1

Inductive task. Now we are in a multicriterial situation. In [2] the authors solve the inductive problem, given a global evaluation, how to learn an objective function which explains global ranking from particular attributes. This is the point where different models have different representations of a utility function.

In [2] an approach is taken, where the global ranking is represented as Choquet integral, and we have to learn the measure. The condition for learning is either;

Approach by the minimization of the quadratic error,

or

Approach based on constraint satisfaction.

For this [2] have to transform data into a more suitable numerical form, cf. Tables 5 and 6.

Grabish and Roubens [1] present an algorithm which specifies a measure such that the Choquet integral mimics the global evaluation. The idea of the first approach (minimization of squared errors) is to identify the fuzzy measure in a Choquet integral: We suppose that the decision maker is able to assess a numerical score for each act and each criterion, and also a numerical global score for each act. So we want to find the fuzzy measure which minimizes the total squared error of the model.

Table 5: Mapping from class and rank to $[0, 1]$

class	*interval for the global score*
good	[0.75, 1.0]
average	[0.4, 0.75]
bad	[0.0, 0.4]

Table 6: Numerical data on criteria and global performance

name	*precision*	*rapidity*	*driving*	*communication*	global
Arthur	1.000	1.000	0.750	0.250	0.133
Lancelot	0.750	0.750	0.750	0.750	0.917
Yvain	1.000	0.625	0.500	1.000	0.833
Perceval	0.250	0.500	0.750	0.750	0.276
Erec	0.375	1.000	0.500	0.750	0.575

name	*precision*	*rapidity*	*driving*	*communication*	global 2nd
Arthur	1.000	1.000	0.750	0.250	0.3
Lancelot	0.750	0.750	0.750	0.750	0.75
Yvain	1.000	0.625	0.500	1.000	0.7
Perceval	0.250	0.500	0.750	0.750	0.35
Erec	0.375	1.000	0.500	0.750	0.5

In the second approach (constrained satisfaction) we assume that we have an expert who is able to tell the relative importance of criteria and kind of interaction between them, if any. All this information can be transformed in terms of linear equalities or inequalities linking the unknown weights. These methods are in fact not comparable, since they do not take exactly the same input, nor provide the same kind of output.

Deductive task. In a similar setting, having trainees and their achievements (same data) we can assume that from previous experiments we already have a utility function. Now the problem is about efficient algorithms to find the best trainee, assuming we have a huge set of data, possibly distributed, and so the question of efficiency becomes crucial.

In this paper we describe the problem setting which is a common starting point for different approaches.

3 Preference structures and fuzzy preference structures and their applications

Let us turn our attention to motivation example. We denote by $M = \{A, E, L, P, Y\}$ the set of all trainees. We are able to construct the large preference relations R_P, R_R, R_D and R_C derived from orderings in our four criteria (precision, rapidity, driving, communication):

R_P	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	1	0
L	0	1	1	1	0
P	0	0	0	1	0
Y	1	1	1	1	1

$R_P = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,Y],[Y,A],[A,L],[A,E],[A,P],[Y,L],[Y,E],[Y,P],[L,E],[L,P],[E,P]\}$

R_R	A	E	L	P	Y
A	1	1	1	1	1
E	1	1	1	1	1
L	0	0	1	1	1
P	0	0	0	1	0
Y	0	0	0	1	1

$R_R = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,E],[E,A],[A,L],[A,Y],[A,P],[E,L],[E,Y],[E,P],[L,Y],[L,P],[Y,P]\}$

R_D	A	E	L	P	Y
A	1	1	1	1	1
E	0	1	0	0	1
L	1	1	1	1	1
P	1	1	1	1	1
Y	0	1	0	0	1

$R_D = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[A,L],[L,A],[A,P],[P,A],[L,P],[P,L],[A,E],[A,Y],[L,E],[L,Y],[P,E],[P,Y],[E,Y],[Y,E]\}$

R_C	A	E	L	P	Y
A	1	0	0	0	0
E	1	1	1	1	0
L	1	1	1	1	0
P	1	1	1	1	0
Y	1	1	1	1	1

$R_C = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[Y,L],[Y,P],[Y,E],[Y,A],[L,P],[P,L],[L,E],[E,L],[P,E],[E,P],[L,A],[P,A],[E,A]\}$

And we are able to construct large preference relation R_I which is derived from instructor's ordering, too:

R_I	A	E	L	P	Y
A	1	0	0	1	0
E	1	1	0	1	0
L	1	1	1	1	1
P	1	0	0	1	0
Y	1	1	1	1	1

$R_I = \{[A,A],[E,E],[L,L],[P,P],[Y,Y],[L,Y],[Y,L],[L,E],[L,A],[L,P],[Y,E],[Y,A],[Y,P],[E,A],[E,P],[A,P],[P,A]\}$

The relation R_I is partialy order set. For global evaluation we will extend this partial ordering to linear ordering. First, we need order the criteria.

The first idea is: we can pairwise compare the relations R_P, R_R, R_D and R_C with relation R_I by the following rule: $\frac{|X \cap R_I|}{|X \triangle R_I|}$, where $X \in \{R_P, R_R, R_D, R_C\}$. This method gives the following ordering of criteria: communication >precision >rapidity > driving. The global evaluation optained from instructor's ordering and previos ordering of criteria is:

$$Y > L > E > P > A$$

In general, ordering of criteria we optaine from relation preference, which is given by this equality $P = R \cap co(R^t)$. However, in this example we have got the same ordering of criteria via both relations with respect to previous method for comparing the relations.

P_P	A	E	L	P	Y
A	0	1	1	1	0
E	0	0	0	1	0
L	0	1	0	1	0
P	0	0	0	0	0
Y	0	1	1	1	0

P_R	A	E	L	P	Y
A	0	0	1	1	1
E	0	0	1	1	1
L	0	0	0	1	1
P	0	0	0	0	0
Y	0	0	0	1	0

$P_P = \{[A,L],[A,E],[A,P],[Y,L],[Y,E],[Y,P],[L,E],[L,P],[E,P]\}$

$P_R = \{[A,L],[A,Y],[A,P],[E,L],[E,Y],[E,P],[L,Y],[L,P],[Y,P]\}$

P_D	A	E	L	P	Y
A	0	1	0	0	1
E	0	0	0	0	0
L	0	1	0	0	1
P	0	1	0	0	1
Y	0	0	0	0	0

$P_D = \{[A,E],[A,Y],[L,E],[L,Y],[P,E],[P,Y]\}$

P_C	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	0	0
L	1	0	0	0	0
P	1	0	0	0	0
Y	1	1	1	1	0

$P_C = \{[Y,L],[Y,P],[Y,E],[Y,A],[L,A],[P,A],[E,A]\}$

P_I	A	E	L	P	Y
A	0	0	0	0	0
E	1	0	0	1	0
L	1	1	0	1	0
P	0	0	0	0	0
Y	1	1	0	1	0

$P_I = \{[L,E],[L,A],[L,P],[Y,E],[Y,A],[Y,P],[E,A],[E,P]\}$

Fuzzification. For better expresion of reality, we can use fuzzy preference structures. One of possible approaches of fuzzification of preference relations from our motivation example is given in next tables:

FP_P	A	E	L	P	Y
A	0	$\frac{2}{3}$	$\frac{1}{3}$	1	0
E	0	0	0	$\frac{1}{3}$	0
L	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0
P	0	0	0	0	0
Y	0	$\frac{2}{3}$	$\frac{1}{3}$	1	0

FP_R	A	E	L	P	Y
A	0	0	$\frac{1}{3}$	1	$\frac{2}{3}$
E	0	0	$\frac{1}{3}$	1	$\frac{2}{3}$
L	0	0	0	$\frac{2}{3}$	$\frac{1}{3}$
P	0	0	0	0	0
Y	0	0	0	$\frac{1}{3}$	0

FP_D	A	E	L	P	Y
A	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$
E	0	0	0	0	0
L	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$
P	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$
Y	0	0	0	0	0

FP_C	A	E	L	P	Y
A	0	0	0	0	0
E	$\frac{1}{3}$	0	0	0	0
L	$\frac{1}{3}$	0	0	0	0
P	$\frac{1}{3}$	0	0	0	0
Y	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0

FP_I	A	E	L	P	Y
A	0	0	0	0	0
E	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0
L	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	0
P	0	0	0	0	0
Y	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0

It is not difficult to see, that fuzzy preference structure gives the following ordering of our criteria: communication>precision>rapidity=driving. We can see, this ordering is different from ordering, which we obtain via strict preference structure, however it still has no influence on ordering of trainees. Note, that another fuzzification leeds to different ordering of criteria and subsequently to different ordering of trainees.

Acknowledgments

Supported by Project 1ET100300517 of the Program "Information Society".

References

[1] M. Grabish and M. Roubens. *An axiomatic approach of interaction in multicriteria decision making.* In 5th Eur. Congr. on Intelligent Techniques and Soft Computing (EUFIT 97), Aachen, Germany, 1997.

[2] M. Grabish and M. Roubens. *Application of the Choquet integral in multicriteria decision making.* In M. Grabisch, T. Murofushi, and M. Sugeno, editors, Fuzzy Measures and Integrals - Theory and Applications, pages 348–374. Physica Verlag, 2000.

[3] T. Horváth, S. Krajči, R. Lencses, P. Vojtáš. *An ILP model for a monotone graded classification problem.* Kybernetika 40,3 (2004), 317–332.

[4] S. Krajči, R. Lencses, & P. Vojtáš. *A comparison of fuzzy and annotated logic programming*, Fuzzy Sets and Systems, 144 (2004), 173–192.