Consistency of friction type condition to the Navier-Stokes equations in modeling of water flow in vaneless machines

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Abstract

The contribution deals with modeling of flow in vaneless machines. The Navier-Stokes system of equations for incompressible liquid is used. Due to different roughness of the machine inner surfaces a special boundary condition of friction type is proposed. To show its consistency to the model, the weak formulation of stationary problem is derived and existence of the solution is proved.

1 Introduction

Vaneless machines powered by power water or pressure air are used in various areas of technology. Let us mention rotating washing brush, drilling machines, small water power stations etc. Although the efficiency of these machines is not too high they became popular for their simplicity, reliability and low price.

Principle of the vaneless machines is the following. The rotor on a shaft in bearings is axially symmetric, usually its surface is a part of a cone or a cylinder. The working medium flows around the rotor such that trajectories of its particles are screw curves. Due to viscosity the fluid transfers part of its kinetic energy to the rotor. To increase the efficiency the surface of the rotor is rough while the surface of the static parts is smooth.

Besides materials of MiRiS and SETUR company there are few materials dealing with flow in vaneless machines. In [5] J. Pelant set up the model and carried out numerical simulations for machines driven by pressure air. For compressible fluid the model uses the non-slip condition.

In the contribution a model of water flow in the machine is proposed. The differential equations are completed with a special friction type boundary condition. They are inspired by similar condition in [2]. To show consistency of this condition to the Navier-Stokes system of equations the weak formulation of the problem is derived and existence of the solution is proved.

2 Selection of the model

We shall confine modeling to working space of the machine around conic surface of rotor where the flow of working medium transfers part of its kinetic energy to the rotor. Since we shall deal with vaneless machines driven by power water, we adopt assumption of incompressible fluid flow.

The simplest model would assume inviscid flow. But in the vaneless machines due to axial symmetry, the pressure of the flowing medium cannot transfer its kinetic energy to the rotor in contrast to classical motors with vanes, thus model of ideal inviscid fluid cannot be used and viscosity of the medium must be assumed. Motion of viscous fluid is modeled by the Navier-Stokes system of equations.

For viscous liquids the non-slip condition on the wall is usually prescribed. The condition requires zero diference between velocity of liquid $\boldsymbol{u}$ on the liquid surface and velocity $\boldsymbol{u}_b$ of the wall: $\boldsymbol{u}-\boldsymbol{u}_b=0$. Nevertheless, such condition pays no attention to the roughness of the wall which plays important role in the vaneless machines and thus is unrealistic.

To describe this situation we propose a friction type boundary condition $\boldsymbol{T_t}=-\mu_0\cdot(\boldsymbol{u}-\boldsymbol{u}_b)$ saying that the tangent stress $\boldsymbol{T_t}$ of liquid acting on the wall is proportional to difference of the liquid velocity and velocity of the wall. This linear dependence can be generalized to nonlinear dependence with a continuous positive function $g(\xi)$

$$\boldsymbol{T_t}=-g(|\boldsymbol{u}-\boldsymbol{u}_b)|)\cdot(\boldsymbol{u}-\boldsymbol{u}_b)\,. \tag{1}$$

To show that this condition is consistent with the model, we shall study existence of the weak solution.

3 System of differential equations

Motion of viscous incompressible liquid is described by the the Navier-Stokes system of equations, see e.g. [1],[3],[6]. We shall use the Einstein convention on summation over repeated indices, i.e. $f_j u_j$ means $\sum_{j=1}^{3} f_j u_j$. In the orthogonal coordinates $\boldsymbol{x}=(x_1,x_2,x_3)$ the unknown velocity vector function $\boldsymbol{u}=(u_1,u_2,u_3)$ and scalar pressure function p satisfy the scalar continuity equation and the vector balance equation

$$\frac{\partial u_i}{\partial x_i}=0\,, \qquad \frac{\partial u_i}{\partial t}+u_j\,\frac{\partial u_i}{\partial x_j}=\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}+f_i\,, \tag{2}$$

where ρ denotes the constant density. The stress tensor $\boldsymbol{\tau}=(\tau_{ij})$ is connected with the velocity by the relation $\boldsymbol{\tau}=(-p+\lambda\,\mathrm{div}\,\boldsymbol{u})\boldsymbol{I}+\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top\right)$ with constants λ,μ. For incompressible liquid due to the continuity equation the relation is simplified to

$$\tau_{ij}=-p\,\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\,. \tag{3}$$

Using the preceding relation we obtain vector equation ($\nu=\mu/\rho$)

$$\frac{\partial u_i}{\partial t}+u_j\,\frac{\partial u_i}{\partial x_j}=\nu\sum_{j}^{3}\frac{\partial^2 u_i}{\partial x_j^2}-\frac{\partial p}{\partial x_i}+f_i\,. \tag{4}$$

We shall deal with the space between static surface of the machine denoted by Γ_{ws} and rotating axis symmetric conic surface of the rotor denoted by Γ_{wr}. Further the modeled volume of liquid denoted by Ω is bounded by inlet surface Γ_{in} and the outlet surface Γ_{out}. The unit normal vector to the boundary $\partial\Omega$ of the domain will be denoted by $\boldsymbol{n}=(n_1,n_2,n_3)$.

On the inlet part of the boundary Γ_{in} the velocity vector is prescribed $\boldsymbol{u}=\boldsymbol{u}_b$. On the outlet part of the boundary Γ_{out} zero normal force $\tau_{ij}n_j=0$ and the pressure $p=p_{out}$ can be prescribed. On the walls $\Gamma_w=\Gamma_{ws}\cup\Gamma_{vr}$ zero normal component of velocity $\boldsymbol{u}\cdot\boldsymbol{n}=0$ and the friction condition (1) is prescribed.

To show that the proposed condition (1) on Γ_w is consistent to the Navier-Stokes equations we shall study existence of the weak solution. The introduced outlet condition causes difficulties in proof of coercivity even with the classical non-slip condition $\boldsymbol{u}=\boldsymbol{u}_b$ on Γ_w. Thus for proof of existence of solution on Γ_{out} we replace that condition by the same condition $\boldsymbol{u}=\boldsymbol{u}_b$ as on Γ_{in}. Thus vector function $\boldsymbol{u}_b$ is defined on the whole $\partial\Omega$. Summary of the boundary conditions yields condition (1) on Γ_w and

$$\boldsymbol{u}=\boldsymbol{u}_b \quad \text{on} \ \ \Gamma_{in}\cup\Gamma_{out}\,, \qquad \boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{on} \ \ \Gamma_w\,. \tag{5}$$

4 Integral identity

To derive integral identity we multiply the i-th balance equation with the i-th component v_i of a test vector function $\boldsymbol{v} = (v_1, v_2, v_3)$, and integrate over domain Ω the sum over i

$$\int_\Omega \frac{\partial u_i}{\partial t}\, v_i \,\mathrm{d}\boldsymbol{x} + \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \,\mathrm{d}\boldsymbol{x} = \nu \int_\Omega \frac{\partial^2 u_i}{\partial x_j^2}\, v_i \,\mathrm{d}\boldsymbol{x} - \int_\Omega \frac{\partial p}{\partial x_i}\, v_i \,\mathrm{d}\boldsymbol{x} + \int_\Omega f_i\, v_i \,\mathrm{d}\boldsymbol{x}\,.$$

We integrate two integrals by parts

$$\int_\Omega \frac{\partial^2 u_i}{\partial x_j^2}\, v_i \,\mathrm{d}\boldsymbol{x} = \int_\Gamma \frac{\partial u_i}{\partial x_j}\, v_i\, n_j \,\mathrm{d}S - \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \,\mathrm{d}\boldsymbol{x}\,, \tag{6}$$

$$\int_\Omega \frac{\partial p}{\partial x_i}\, v_i \,\mathrm{d}\boldsymbol{x} = \int_\Gamma p\, v_i\, n_i \,\mathrm{d}S - \int_\Omega p \frac{\partial v_i}{\partial x_i} \,\mathrm{d}\boldsymbol{x}\,. \tag{7}$$

Since the velocity vector $\boldsymbol{u}$ should satisfy the continuity equation, the test function $\boldsymbol{v}$ is supposed to satisfy $\operatorname{div} \boldsymbol{v} = 0$ in Ω as well and the last integral in (7) vanishes. Due to boundary conditions (5) we choose also $v_i = 0$ on $\Gamma_{in} \cup \Gamma_{out}$ and $v_i n_i = 0$ on Γ_w. Thus the integral vanishes over $\partial\Omega$ in (7) and the surface integral of (6) vanishes over $\Gamma_{in} \cup \Gamma_{out}$.

Friction condition on the wall. On the boundary Γ_w the stress vector equals to $\boldsymbol{T} = \boldsymbol{\tau}\cdot\boldsymbol{n}$, i.e. $(\boldsymbol{T})_i = \tau_{ij} n_j$. Using (3) with a tangent vector $\boldsymbol{t}$ and boundary condition (1) we obtain

$$\boldsymbol{T}\cdot\boldsymbol{t} = \tau_{ij} n_j t_i = \mu \frac{\partial u_i}{\partial x_j} n_j t_i = -g(|\boldsymbol{u} - \boldsymbol{u}_b|)(u_i - u_{bi}) t_i$$

since $\delta_{ij} n_j t_i = 0$ and $\frac{\partial u_j}{\partial x_i} n_j t_i = 0$ due to $u_j n_j = 0$ around Γ_w. Thus for test function $\boldsymbol{v}$ satisfying $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ_w we obtain

$$\int_{\Gamma_w} \frac{\partial u_i}{\partial x_j}\, v_i\, n_j \,\mathrm{d}S = -\int_{\Gamma_w} g(|\boldsymbol{u} - \boldsymbol{u}_b|)(u_i - u_{bi}) v_i \,\mathrm{d}S\,.$$

Denoting the scalar product in $L^2(\Omega, \boldsymbol{R}^3)$ by $(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega u_i v_i \,\mathrm{d}\boldsymbol{x}$ and

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \,\mathrm{d}\boldsymbol{x}\,, \qquad b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega u_j \frac{\partial u_i}{\partial x_j}\, v_i \,\mathrm{d}\boldsymbol{x}\,,$$

$$\langle G(\boldsymbol{u} - \boldsymbol{u}_b), \boldsymbol{v}\rangle = \int_{\Gamma_w} g(|\boldsymbol{u} - \boldsymbol{u}_b|)(u_i - u_{bi})\, v_i \,\mathrm{d}S$$

we obtaine the following integral identity

$$\left(\frac{\partial u_i}{\partial t}, v_i\right) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + \langle G(\boldsymbol{u} - \boldsymbol{u}_b), \boldsymbol{v}\rangle = (\boldsymbol{f}, \boldsymbol{v})\,. \tag{8}$$

5 The stationary problem

According to the continuity equation and boundary conditions (5) we choose the following Sobolev space

$$H_{div} = \left\{\boldsymbol{v} \in H^{1,2}(\Omega, \boldsymbol{R}^3) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\right\}\,,$$

a space for test functions $\boldsymbol{v}$ and a linear set for solution $\boldsymbol{u}$

$$V_0 = \{\boldsymbol{v} \in H_{div} \mid \boldsymbol{v} = 0 \text{ on } \Gamma_{in} \cup \Gamma_{out}, \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_w\}\,,$$

$$V_b = \{\boldsymbol{u} \in H_{div} \mid \boldsymbol{u} = \boldsymbol{u}_b \text{ on } \Gamma_{in} \cup \Gamma_{out}, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_w\}\,.$$

Now we can define *weak solution* to the stationary problem:

Vector function $\boldsymbol{u} \in V_b$ *is called weak solution to the stationary problem if*

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \langle G(\boldsymbol{u} - \boldsymbol{u}_b), \boldsymbol{v}\rangle = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V_0 \,.$$

The solution $\boldsymbol{u}$ is in set V_b which is not a linear space. To enable using of abstract existence theorem we translate the problem by $\boldsymbol{u} := \boldsymbol{u}^* + \boldsymbol{U}$ using a vector function $\boldsymbol{U} \in V_b \subset H_{div}$ i.e. satisfying (5).

Then we look for $\boldsymbol{u}^* + \boldsymbol{U}$, such that $\boldsymbol{u}^* \in V_0$ and for all $\boldsymbol{v} \in V_0$

$$a(\boldsymbol{u}^* + \boldsymbol{U}, \boldsymbol{v}) + b(\boldsymbol{u}^* + \boldsymbol{U}, \boldsymbol{u}^* + \boldsymbol{U}, \boldsymbol{v}) + \langle G(\boldsymbol{u}^*), \boldsymbol{v}\rangle = \boldsymbol{f}(\boldsymbol{v}) \,. \tag{9}$$

Existence of the solution can be proven using the following theorem, see [4]:

THEOREM *Let* V *be a separable reflexive Banach space and* $A : V \to V^*$ *an operator which is*

— *weakly continuous, i. e.* $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \Rightarrow A(\boldsymbol{v}_n) \rightharpoonup A(\boldsymbol{v})$,

— *and coercive, i. e.* $\lim_{\|\boldsymbol{u}\| \to \infty} \frac{\langle A(\boldsymbol{u}), \boldsymbol{u}\rangle}{\|\boldsymbol{u}\|} = \infty$.

Then the equation $A(\boldsymbol{u}) = F$ *admits a solution for any* $F \in V^*$.

The proof of existence of solution to the stationary problem consist in verifying the assumptions of the theorem by means of e.g. results of [4].

Denoting paring of spaces V_0^* and V_0 by $\langle \cdot, \cdot \rangle$, the identity (9) can be rewritten to the operator equation $A(\boldsymbol{u}) = F$ by setting

$$\langle A(\boldsymbol{u}), \boldsymbol{v}\rangle = a(\boldsymbol{u}^*, \boldsymbol{v}) + b(\boldsymbol{u}^*; \boldsymbol{u}^*, \boldsymbol{v}) + b(\boldsymbol{u}^*; \boldsymbol{U}, \boldsymbol{v}) + b(\boldsymbol{U}; \boldsymbol{u}^*, \boldsymbol{v}) + +\langle G(\boldsymbol{u}^*), \boldsymbol{v}\rangle$$

$$\langle F, \boldsymbol{v}\rangle = (\boldsymbol{f}, \boldsymbol{v}) - a(\boldsymbol{U}, \boldsymbol{v}) - b(\boldsymbol{U}; \boldsymbol{U}, \boldsymbol{v}) \,.$$

The space V_0 is a separable reflexive Banach space. Continuous bilinear forms are weakly continuous. Due to compact imbedding $W^{1,2}(\Omega) \subset\subset L^4(\Omega)$ weak convergence of u_i in $W^{1,2}(\Omega)$ yields strong convergence of u_i in $L^4(\Omega)$ and weak continuity of trilinear forms follow since they are linear in ∇u.

Concerning coercivity the leading bilinear form is elliptic and the other terms do not violate this coercivity. Indeed, the "new" term $\langle G(\boldsymbol{u}^*), \boldsymbol{u}\rangle$ is nonnegative, $b(\boldsymbol{u}^*; \boldsymbol{u}^*, \boldsymbol{u}^*) = 0$, see e.g. [3], and by a special choice of "small" $\boldsymbol{U} \in V_b$ constructed by means of a cut off function, see e.g. [3], also the remaining trilinear forms with $\boldsymbol{U}$ can be made arbitrary small and the result follows.

6 Conclusion

For modeling of the flow in vaneless machines we proposed a new friction type boundary condition for incompressible viscous liquid. The condition contains the continuous nonnegative constitutive function $g(\xi)$. The case $g = \nu_0$ yields linear dependence, $g = 0$ free surface of the liquid and the limit $g = \infty$ the non-slip condition.

In the contribution we proved existence of the weak solution to the problem, thus the proposed condition seems to be consistent with the Navier-Stokes system of equations.

Let us mention that using technique introduced in [3] also existence of the weak solution to the corresponding evolution initial boundary value problem can be proved by Rothe functions technique.

Role of the proposed friction type condition will be further studied by numerical experiments and comparison to the real vaneless machines.

References

[1] P.Constantin, C. Foias. *Navier-Stokes equations*, Univ. of Chicago Press, 1988.

[2] G. Duvaut, J. L. Lions. *Les inquations en mchanics et en physique,* Dunot, 1972.

[3] M. Feistauer. *Mathematical methods in fluid dynamics*, Pitman Monographs and Surveys in Pure and Applied Mathematics 67, Longman Scientific and Technical, Harlow, 1993.

[4] J. Franců. Weakly continuous operators, Applications to differential equations, *Application of Mathematics*, 39 (1994), no 1, 45–56.

[5] J. Pelant, M. Kyncl. The Navier-Stokes equations and boundary value conditions for some 3D D axis-symmetrical flows, *MiRiS – SETUR company materials*, Prague, 2005.

[6] R. Temam. *Navier Stokes equations*, North-Holland, 1977.