On some applications of Lyapunov functions

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It is well known that the Lyapunov's second method (see[1-5]) is an interesting and fruitful technique that has gained increasing significance and has given decisive impetus for modern development of a stability theory of differential and integrodifferential equations. Lyapunov's functions serve as a vehicle to transform given complicated systems into relatively simpler systems, and therefore, it is enough to investigate the properties of these simpler systems.

In this paper we will investigate using Lyapunov functions an asymptotic behaviour of solutions the following initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0, \ t_0 \geq 0, \tag{1}$$

where $f \in C[R^+ \times s(\rho),\ R^n],\ s(\rho) = \{x \in R^n : |x| < \rho\}$. Assume, for convenience, that the solution $x(t) = x(t, t_0, x_0)$ of (1) exists, is unique for $t \geq t_0$ and $f(t, 0) = 0$ so that we have the trivial solution $x = 0$.

Denote $\kappa = \{\sigma \in C[[0, \rho), R^+]\}$ where $\sigma(t)$ is strictly increasing and $\sigma(0) = 0$.

A function $V \in C^1[R^+ \times s(\rho), R^+]$ will be called positive definite (descrescent) if there exists a function $a \in \kappa$ such that $V(t, x) \geq a(|x|)$ $(V(t, x) \leq a(|x|))$.

Now let us state the well known original theorems of Lyapunov for stability and asymptotic stability in a suitable form.

Theorem 1. *Assume that*

(H) $V \in C^1[R^+ \times s(\rho),\ R^+],\ V$ *is positive definite and* $V(t, 0) \equiv 0$.

If $V'(t, x) \leq 0$ *on* $R^+ \times s(\rho)$, *then the trivial solution of (1) is stable.*

Theorem 2. *Suppose that condition (H) holds. Assume further that V is decrescent and* $V'(t, x) \leq -c(|x|)$ *on* $R^+ \times s(\rho)$, *where* $c \in \kappa$. *Then the trivial solution of (1) is uniformly asymptotically stable.*

These two theorems have been modified, extended and generalized in various aspects. For example , if we omit that V is descrescent in Theorem 2. and suppose that $V(t, 0) \equiv 0$, we still get the stability of the trivial solution (see[3]).
If we suppose that condition (H) holds, f is bounded on $R^+ \times s(\rho)$, and $V'(t, x) \leq -c(|x|)$ on $R^+ \times s(\rho)$,, where $c \in \kappa$ then the trivial solution of (1) is asymptotically stable (The Marachkov's theorem [4]).
The positive definiteness of $V(t, x)$ in the Marachkov's theorem can be weakened as follows.
If we assume that , instead of the positive definiteness of $V(t, x)$, a weaker condition, namely,

$V(t,x) \equiv 0,\ V(t,x) \geq 0$, then the conclusion of the Marachkov's theorem holds.
Now we give an generalization of the Marachkov's result using two Lyapunov functions.

Theorem 3. *Assume*

(i) $V \in C^1[R^+ \times s(\rho), R^+],$ *V is positive definite,* $V(t,0) \equiv 0$ *and* $V'(t,x) \leq -c(W(t,x))$ *on* $R^+ \times s(\rho)$, *where* $c \in \kappa$.

(ii) $W \in C^1[R^+ \times s(\rho),\ R^+]$, *W is positive definite and* $W'(t,x)$ *is bounded from above or from below on* $R^+ \times s(\rho)$,.

Then the trivial solution of (1) is asymptotically stable.

The first Lyapunov function V serves to obtain the stability and the second Lyapunov function W relates suitably to the first one. The advantage is that one can utilize the monotone character of $V(t,x(t))$.

Acknowledgement

This research has been supported by the Czech Ministry of Education in the frame of MSM002160503 Research Intention MIKROSYN New Trends in Microelectronic Systems and Nanotechnologies.

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