Using the Ważewski's Topological Method for Functional Differential Equations with Unbounded Delay

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The investigation of the asymptotic behavior of solutions of differential equations is often based on using the Ważewski's topological method.

The necessary condition for applying this method is the continuous dependence of solutions on initial data. This property RDFE with unbounded delay in the sense given in [1] is known. Let us shortly describe this notion.

In the theory of RFDE's with infinite delay the symbol y_t is defined as follows:
Let $t_0 \geq 0$, $A > 0$ be real constants and $y \in C[(-\infty, t_0+A], \mathbb{R}^n)$. For each $t \in [t_0, t_0+A)$, we define y_t by $y_t(\vartheta) = y(t+\vartheta),\ -\infty < \vartheta \leq 0$ and write $y_t \in \mathcal{C} \equiv C[(-\infty, 0], \mathbb{R}^n]$.

The system *retarded functional differential equations with infinite delay* (RFIDE's) is defined as

$$\dot{y}(t) = f(t, y_t) \tag{1}$$

where $f \in C([t_0, t_0+A) \times \mathcal{C}, \mathbb{R}^n)$. The function $y \in C[(-\infty, t_0+A), \mathbb{R}^n] \cap C^1[[t_0, t_0+A), \mathbb{R}^n]$ satisfying (1) on $[t_0, t_0+A)$ is called a *solution of system* (1) *on* $[t_0, t_0+A)$. In this paper we use the norm $\|\psi\| = \sup\limits_{-\infty<t\leq 0} e^{p(t)}|\psi(t)|$ for the functions $\psi \in \mathcal{B} \subset \mathcal{C}$ such that $|\psi(t)| = o(e^{p(t)})$, for $t \to -\infty$. Here, $p(t)$ is any fixed continuous function such that $p(t) - p(t_0+t)$ is bounded above for fixed $t_0 > 0$ and $-\infty < t \leq 0$. This choice of the phase space $\mathcal{B}$ enables as to assign sufficient conditions, which guarantee the existence, uniqueness and continuous dependence of the solutions in the usual form. (see [1, p. 50]).

If $(t_0, \phi) \in \Omega \subseteq [t_0, t_0+A) \times \mathcal{B}$, then *there exists a solution* $y = y(t_0, \phi)$ of the system RFDE's (1) through (t_0, ϕ) (see [1, p. 50]). Moreover this *solution is unique* (see [1, p. 54]) if there exist constants δ, γ and a nonnegative function $g:\ [0, \delta] \to [0, \infty)$ continuous at $t = 0$ and $g(0) = 0$ such that for any $x, y \in A(t_0, \phi, \delta, \gamma)$ holds

$$\left| \int_{t_0}^{t} (f(s, x_s) - f(s, y_s))ds \right| \leq g(t-t_0) \sup_{t_0 \leq s \leq t} |x(s) - y(s)|, \tag{2}$$

where

$$A(t_0, \phi, \delta, \gamma) = \{x : (\infty, t_0] \to \mathbb{R}^n, x_{t_0} = \phi, \sup_{t_0 \leq t \leq t_0+\delta} |x(t) - \phi(0)| \leq \gamma\}.$$

Then the property of the *continuous dependence* holds too (see [1, p. 63]), i.e. for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $(s, \psi) \in \Omega$, $|s - t_0| < \delta$ and $\|\psi - \phi\| < \delta$ imply

$$\|y_t(s, \psi) - y_t(t_0, \phi)\| < \epsilon, \text{ for all } t \in [\zeta, A]$$

where $y(s, \psi)$ is the solution of the system (1) through (s, ψ), $\zeta = \max\{s, t_0\}$ (see [1, p. 63]).

Let $l_i,\ m_j,\ i = 1, \ldots p,\ j = 1, \ldots s,\ p + s > 0$ be real-valued C^1-functions defined on $\mathbb{R} \times \mathbb{R}^n$. The set

$$\tilde{\omega}_{t^*} = \{(t, y) \in [t^*, \infty) \times \mathbb{R}^n,\ l_i(t, y) < 0,\ m_j(t, y) < 0,\ \text{for all } i, j\ \}$$

will be called a *polyfacial set*. Definition of the polyfacial set $\tilde{\omega}_{t^*}$ *regular with respect to Eq.* (1) is analogous with RFDE [3].

Theorem 1 *Let $p > 0$. Let $\tilde{\omega}_{t^*}$ be a nonempty polyfacial set, regular with respect to Eq.* (1)*, let the function $f \in C(\tilde{\Omega}, \mathbb{R}^n)$ be completely continuous and constants δ, γ and a nonnegative function $g :\ [0, \delta] \to [0, \infty)$ continuous at $t = 0$ and $g(0) = 0$ such that for any $x, y \in A(t_0, \phi, \delta, \gamma)$ holds (2),*

$$W = \{(t, y) \in \partial\tilde{\omega}_{t^*} :\ m_j(t, y) < 0,\ j = 1, \ldots, s\}. \tag{3}$$

Let Z be a subset of $\tilde{\omega}_{t^*} \cup W$ and let mapping $q : B = \overline{Z} \cap (Z \cup W) \to \mathcal{C}$ be continuous and such that if $z = (\delta, y) \in B$, then $(\delta, q(z)) \in \tilde{\Omega}$, and :

1) *If $z \in Z \cap \tilde{\omega}$, then $(\delta + \vartheta, q(z)(\delta + \vartheta)) \in \tilde{\omega}$ for $\vartheta \in (-\infty, 0]$,*

2) *If $z \in W \cap B$, then $(\delta, q(z)(\delta)) = z$ and $(\delta + \vartheta, q(z)(\delta + \vartheta)) \in \tilde{\omega}$ for $\vartheta \in (-\infty, 0)$.*

Let, moreover, $Z \cap W$ be a retract of W, but not a retract of Z. Then there exists a $z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}$ such that $(t, y(\delta_0, q(z_0))(t)) \in \tilde{\omega}$ for every $t \in D_{\delta_0, q(z_0)}$.

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