Using the Ważewski's Topological Method for Functional Differential Equations with Unbounded Delay

Josef Diblík, Zdeněk Svoboda

Brno University of Technology, Brno, Czech Republic
E-mail address: diblik@feec.vutbr.cz, diblik.j@fce.vutbr.cz

Brno University of Technology, Faculty of Electrical Engineering and Communication, Brno, Czech Republic
E-mail address: svobodaz@feec.vutbr.cz

Keywords: Ważewski's Topological Method,Functional Differential Equations with Unbounded Delay
MOS Subject Classification: 34K15, 34K25.

The investigation of the asymptotic behavior of solutions of differential equations is often based on using the Ważewski's topological method. This principle was modified by Rybakowski in 1980 for retarded functional differential equations (RFDE) in [6].

At first this principle is formulated for the system of curves generally. The system solutions with initial value ϕ, (such that the functions ϕ remain in a polyfacial set) is interpretable as the system curves and the topological principle is applicable for the solutions of RFDE's too.

The necessary condition for applying this method is the continuous dependence of solutions on initial data.

This property RDFE with unbounded delay in the sense given in [2] is known. Let us shortly describe this notion.

In the theory of RFDE's with infinite delay the symbol y_t, which expresses *"taking into account"* the history of the process $y(u)$, for $u \le t$, is defined as follows:
Let $t_0 \ge 0$, $A > 0$ be real constants and $y \in C[(-\infty, t_0+A], \mathbb{R}^n)$. For each $t \in [t_0, t_0+A)$, we define y_t by $y_t(\vartheta) = y(t+\vartheta)$, $-\infty < \vartheta \le 0$ and write $y_t \in \mathcal{C} \equiv C[(-\infty, 0], \mathbb{R}^n]$.

With the aid of symbol y_t the system *retarded functional differential equations with infinite delay* (RFIDE's) is defined as

$$\dot{y}(t) = f(t, y_t) \tag{1}$$

where $f \in C([t_0, t_0+A)\times\mathcal{C}, \mathbb{R}^n)$. The function $y \in C[(-\infty, t_0+A), \mathbb{R}^n]\cap C^1[[t_0, t_0+A), \mathbb{R}^n]$ satisfying (1) on $[t_0, t_0+A)$ is called a *solution of system* (1) *on* $[t_0, t_0+A)$.

In the next we will suppose that every nonempty set of functions $\mathcal{B} \subset \mathcal{C}$ is endowed with such suitable norm $||\cdot||$ that $\mathcal{B}$ forms the Banach space. In this paper we use the norm $\|\psi\| = \sup\limits_{-\infty<t\le 0} e^{p(t)}|\psi(t)|$ for the functions $\psi \in \mathcal{B} \subset \mathcal{C}$ such that $|\psi(t)| = o(e^{p(t)})$, for $t \to -\infty$. Here, $p(t)$ is any fixed continuous function such that $p(t)-p(t_0+t)$ is bounded above for fixed $t_0 > 0$ and $-\infty < t \le 0$. This choice of the phase space $\mathcal{B}$ enables as to assign sufficient conditions, which guarantee the existence, uniqueness and continuous dependence of the solutions in the usual form. (see [2, p. 50]).

If $(t_0, \phi) \in \Omega \subseteq [t_0, t_0+A)\times\mathcal{B}$, then *there exists a solution* $y = y(t_0, \phi)$ of the system RFDE's (1) through (t_0, ϕ) (see [2, p. 50]). Moreover this *solution is unique* (see [2, p. 54]) if there exist constants δ, γ and a nonnegative function $g: \ [0,\delta] \to [0,\infty)$ continuous at $t = 0$ and $g(0) = 0$ such that for any $x, y \in A(t_0, \phi, \delta, \gamma)$ holds

$$\left| \int_{t_0}^{t} (f(s, x_s) - f(s, y_s)) ds \right| \le g(t - t_0) \sup_{t_0 \le s \le t} |x(s) - y(s)|, \tag{2}$$

where

$$A(t_0, \phi, \delta, \gamma) = \{x : (\infty, t_0] \to \mathbb{R}^n, x_{t_0} = \phi, \sup_{t_0 \le t \le t_0 + \delta} |x(t) - \phi(0)| \le \gamma\}.$$

Then the property of the *continuous dependence* holds too (see [2, p. 63]), i.e. for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $(s, \psi) \in \Omega$, $|s - t_0| < \delta$ and $\|\psi - \phi\| < \delta$ imply

$$\|y_t(s, \psi) - y_t(t_0, \phi)\| < \epsilon, \text{ for all } t \in [\zeta, A]$$

where $y(s, \psi)$ is the solution of the system (1) through (s, ψ), $\zeta = \max\{s, t_0\}$ (see [2, p. 63]).

Note that these results can be adapted easily for the case (which will be used in the sequel) when $\Omega \subset \mathbb{R} \times \mathcal{B}$ such that the cross-section $\{(\tilde{t}, \varphi) \in \Omega\}$ is an open set in the cross-section $\{(\tilde{t}, \varphi) \in \mathcal{B}\}$ for every $\tilde{t} \in \mathbb{R}$.

If a set $A \subset \mathbb{R} \times \mathbb{R}^n$ is given, then $\operatorname{int} A$, $\overline{A}$ and ∂A denote, as usual, the interior, the closure, and the boundary of A, respectively.

Definition 1 *Let Λ be a topological space, let a subset $\tilde{\Omega} \subset \mathbb{R} \times \Lambda$ be open in $\mathbb{R} \times \Lambda$, and let x be a mapping, associating with every $(\delta, \lambda) \in \tilde{\Omega}$ a function $x(\delta, \lambda) : D_{\delta,\lambda} \to \mathbb{R}^n$ where $D_{\delta,\lambda}$ is an interval in $\mathbb{R}$. Assume 1 through 3:*

1) $\delta \in D_{\delta,\lambda}$.

2) If $t \in \operatorname{int} D_{\delta,\lambda}$, then there is open neighbourhood $\mathcal{O}(\delta, \lambda)$ of (δ, λ) in $\tilde{\Omega}$ such that $t \in D_{\delta',\lambda'}$ holds for all $(\delta', \lambda') \in \mathcal{O}(\delta, \lambda)$.

3) If (δ', λ'), $(\delta, \lambda) \in \tilde{\Omega}$, and $t' \in D_{\delta',\lambda'}$, $t \in D_{\delta,\lambda}$, then

$$\lim_{(\delta',\lambda',t') \to (\delta,\lambda,t)} x(\delta', \lambda')(t') = x(\delta, \lambda)(t).$$

If all these conditions are satisfied, then $(\Lambda, \tilde{\Omega}, x)$ is called a system of curves *in $\mathbb{R}^n$.*

Studying the proof of Theorem 2.1 in [6, p.119], we get the following formulation of it, suitable for our applications.

Lemma 1 (Retract Principle) *Let $(\Lambda, \tilde{\Omega}, x)$ be a system of curves in $\mathbb{R}^n$. Let $\tilde{\omega}$, W, Z be sets. Assume that conditions* 1 *through* 4 *below hold:*

1) *a) $\tilde{\omega} \subset [p^*, \infty) \times \mathbb{R}^n$ where $p^* \in \mathbb{R}$ and the cross-section $\{(\tilde{t}, y) \in \tilde{\omega}\}$ is an open set for every $\tilde{t} \in [p^*, \infty)$, $W \subset \partial\tilde{\omega}$,*

   b) $Z \subset \tilde{\omega} \cup W$, $Z \cap W$ is a retract of W, but not a retract of Z.

2) *There is a continuous map* q*:* $B \to \Lambda$*, where* $B = \overline{Z} \cap (Z \cup W)$*, such that for any* $z = (\delta, y) \in B$*:* $(\delta, q(z)) \in \tilde{\Omega}$*, and if also* $z \in W$*, then* $x(\delta, q(z))(\delta) = y$*.*

3) *Let* A *be the set of all* $z = (\delta, y) \in Z \cap \tilde{\omega}$ *such that for fixed* $(\delta, y) \in A$ *there is a* $t > \delta$*,* $t \in D_{\delta,q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \tilde{\omega}$*.*

   Assume that for every $z = (\delta, y) \in A$ *there is a* $t(z)$*,* $t(z) > \delta$*, such that:*

   a) $t(z) \in D_{\delta,q(z)}$ *and for all* t*,* $\delta \le t < t(z)$*:* $(t, x(\delta, q(z))(t)) \in \tilde{\omega}$*,*
   b) $(t(z), x(\delta, q(z))(t(z))) \in W$*,*
   c) *For any* $\sigma > 0$*, there is a* $t = t(\sigma, z)$*,* $t(z) < t \le t(z) + \sigma$*, such that* $t \in D_{\delta,q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \overline{\tilde{\omega}}$*.*

4) *For any* $z = (\delta, y) \in W \cap B$*, and all* $\sigma > 0$*, there is a* $t = t(\sigma, z)$*,* $\delta < t \le \delta + \sigma$ *such that* $t \in D_{\delta,q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \tilde{\omega}$*.*

Then there is a $z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}$ *such that for every* $t \in D_{\delta_0,q(z_0)}$ *:*

$$(t, x(\delta_0, q(z_0))(t)) \in \tilde{\omega}.$$

Let $l_i,\ m_j,\ i = 1, \dots p,\ j = 1, \dots s,\ p + s > 0$ be real-valued C^1-functions defined on $\mathbb{R} \times \mathbb{R}^n$. The set

$$\tilde{\omega}_{t^*} = \{(t, y) \in [t^*, \infty) \times \mathbb{R}^n,\ l_i(t, y) < 0,\ m_j(t, y) < 0,\ \text{for all } i, j\ \}$$

will be called a *polyfacial set.* We denote $\tilde{\omega} = \{(t, y) |\, l_i(t, y) < 0,\ m_j(t, y) < 0,\ \text{for all } i, j\ \}$.

Definition 2 *A polyfacial set* $\tilde{\omega}_{t^*}$ *is called* regular with respect to Eq. *(1) if* $\alpha)$, $\beta)$, $\gamma)$ *below hold:*

$\alpha)$ *If* $(t, \phi_t) \in \mathbb{R} \times \mathcal{B}$ *and if* $(t + \vartheta, \phi(t + \vartheta)) \in \tilde{\omega}$ *for all* $\vartheta \in [-\infty, 0)$ *and* $t \ge t^*$*, then* $(t, \phi_t) \in \tilde{\Omega}$*.*

$\beta)$ *For all* $i = 1, \dots, p$*, all* $(t, y) \in \partial\tilde{\omega}_{t^*}$ *for which* $l_i(t, y) = 0$ *and for all* $\phi_t \in \mathcal{C}$ *for which* $\phi_t(0) = y$ *and* $(t + \vartheta, \phi(\vartheta)) \in \tilde{\omega}$ *for all* $\vartheta \in (-\infty, 0)$*, it follows that*

$$Dl_i(t, y) \equiv \sum_{r=1}^{n} \frac{\partial l_i}{\partial y_r}(t, y) \cdot f_r(t, \phi_t) + \frac{\partial l_i}{\partial t}(t, y) > 0.$$

$\gamma)$ *For all* $j = 1, \dots, s$*, all* $(t, y) \in \partial\tilde{\omega}_{t^*}$ *for which* $m_j(t, y) = 0$ *and for all* $\phi_t \in \mathcal{C}$ *for which* $\phi_t(0) = y$ *and* $(t + \vartheta, \phi(\vartheta)) \in \tilde{\omega}$ *for all* $\vartheta \in (-\infty, 0)$*, it follows that*

$$Dm_j(t, y) \equiv \sum_{r=1}^{n} \frac{\partial m_j}{\partial y_r}(t, y) \cdot f_r(t, \phi_t) + \frac{\partial m_j}{\partial t}(t, y) < 0.$$

Theorem 1 *Let* $p > 0$*. Let* $\tilde{\omega}_{t^*}$ *be a nonempty polyfacial set, regular with respect to Eq.* (1)*, let the function* $f \in C(\tilde{\Omega}, \mathbb{R}^n)$ *be completely continuous and constants* δ*,* γ *and a nonnegative function* $g :\ [0, \delta] \to [0, \infty)$ *continuous at* $t = 0$ *and* $g(0) = 0$ *such that for any* $x, y \in A(t_0, \phi, \delta, \gamma)$ *holds (2),*

$$W = \{(t, y) \in \partial\tilde{\omega}_{t^*} :\ m_j(t, y) < 0,\ j = 1, \dots, s\}. \tag{3}$$

Let Z *be a subset of* $\tilde{\omega}_{t^*} \cup W$ *and let mapping* $q : B = \overline{Z} \cap (Z \cup W) \to \mathcal{C}$ *be continuous and such that if* $z = (\delta, y) \in B$*, then* $(\delta, q(z)) \in \tilde{\Omega}$*, and :*

1) *If* $z \in Z \cap \tilde{\omega}$, *then* $(\delta+\vartheta, q(z)(\delta+\vartheta)) \in \tilde{\omega}$ *for* $\vartheta \in (-\infty, 0]$,

2) *If* $z \in W \cap B$, *then* $(\delta, q(z)(\delta)) = z$ *and* $(\delta+\vartheta, q(z)(\delta+\vartheta)) \in \tilde{\omega}$ *for* $\vartheta \in (-\infty, 0)$.

Let, moreover, $Z \cap W$ *be a retract of* W*, but not a retract of* Z*. Then there exists a* $z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}$ *such that* $(t, y(\delta_0, q(z_0))(t)) \in \tilde{\omega}$ *for every* $t \in D_{\delta_0, q(z_0)}$.

Proof. We prove the lemma using Lemma 1. The conditions 1), 2) of Lemma 1 are obviously satisfied. Let us verify the conditions 3) and 4).

Verification of the condition 3): Let $z = (\delta, y) \in A$, and let $t(z)$ be the smallest of all $t \geq \delta$ such that $t \in D_{\delta, q(z)}$ and $(t, y(\delta, q(z))(t)) \notin \tilde{\omega}_{t^*}$. Since $(\delta, y(\delta, q(z))(\delta)) = (\delta, q(z)(\delta))) \in \tilde{\omega}$, it follows that $\delta < t(z) < \infty$. Obviously, $(t(z), y(\delta, q(z))(t(z))) \in \partial\tilde{\omega}_{t^*}$ and moreover for $\delta \leq t < t(z)$ it holds: $(t, y(\delta, q(z))(t)) \in \tilde{\omega}$, hence 3)a) is satisfied.

Let $\phi_t \equiv y_{t(z)}(\delta, q(z))$. Obviously $\phi_t \in \mathcal{C}$. Then $(t(z)), \phi_t) \in \tilde{\Omega}$, and $(t(z), \phi(t(z))) = (t(z), y(\delta, q(z))(t(z))) \in \partial\tilde{\omega}_{t^*}$, and

$$(t+\vartheta, \phi(t+\vartheta)) \in \tilde{\omega}_{t^*}, \quad \text{for } \vartheta \in (-\infty, 0).$$

To prove the condition 3)b) suppose, on the contrary, that $(t(z), \phi(t(z)))) \notin W$. Since $(t(z), \phi(t(z)))) \in \partial\tilde{\omega}_{t^*}$ it follows $m_{j_0}(t(z), \phi(t(z))) = 0$ for some $j_0 \in \{1, \ldots, s\}$. Hence the inequality γ) in Definition 2 is satisfied. Since $y(\delta, q(z))(t)$ is differentiable in t for $t > \delta$, this inequality becomes

$$Dm_{j_0}(t, y(\delta, q(z))(t))|_{t=t(z)} < 0,$$

i.e. for some $\sigma > 0$ and all $0 < h < \sigma$:

$$m_{j_0}(t(z)-h, y(\delta, q(z))(t(z)-h)) > \\ m_{j_0}(t(z), y(\delta, q(z))(t(z))) = m_{j_0}(t(z), \phi(t(z))) = m_{j_0}(t(z), \phi_t(0)) = 0.$$

Hence $(t(z)-h, y(\delta, q(z))(t(z)-h)) \notin \overline{\tilde{\omega_{t^*}}}$. This is a contradiction to 3)a). Then $(t(z), \phi_t(0)) \in W$ and, therefore, 3)b) is satisfied.

It follows that $l_{i_0}(t(z), \phi(t(z), 0)) = 0$ for some $i_0 \in \{1, \ldots, p\}$. Applying β) of Definition 2, we get

$$Dl_{i_0}(t, y(\delta, q(z))(t))\Big|_{t=t(z)} > 0,$$

hence, for some $\sigma > 0$ and all $0 < h < \sigma$:

$$l_{i_0}(t(z)+h, y(\delta, q(z))(t(z)+h)) > l_{i_0}(t(z), y(\delta, q(z))(t(z))) = l_{i_0}(t(z), \phi(t(z))) = 0.$$

Hence $(t(z)+h, y(\delta, q(z))(t(z)+h)) \notin \overline{\tilde{\omega_{t^*}}}$ and 3)c) is satisfied.

Verification of the condition 4): If $z = (\delta, y) \in W \cap B$, then there is a $i_0 \in \{1, \ldots, p\}$ such that $l_{i_0}(\delta, y) = 0$. Let $\phi = q(z)$, then $(\delta+\vartheta, \phi(\delta+\vartheta)) \in \tilde{\omega}$, for all $\vartheta \in (-\infty, 0)$. Hence, the derivative from the right

$$Dl_{i_0}(t, y(\delta, q(z))(t))\Big|_{t=\delta+0} > 0.$$

This implies the existence of some $\sigma > 0$ such that for all $0 < h < \sigma$:

$$l_{i_0}(\delta+h, y(\delta, q(z))(\delta+h)) > l_{i_0}(\delta, y(\delta, q(z))(\delta)) = l_{i_0}(\delta, \phi(\delta)) = 0,$$

i.e. $(\delta+h, y(\delta, q(z))(\delta+h)) \notin \overline{\tilde{\omega}}$ for $0 < h < \sigma$. So, condition 4) of Lemma 1 holds and the Lemma 1 is valid in the described situation. From its conclusion, the conclusion of Lemma 1 follows.

Acknowledgment

This investigation was supported by the Grant A 1163401 of the Grant Agency of the AS CR (Prague) and by the Council of Czech Government MSM 00216 30503.

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