Positive solutions of two-dimensional linear differential delayed systems

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In this contribution we consider system of two linear homogeneous differential equations with variable time lag

$$\dot{y}_1(t) = -a(t)y_2(t-\tau(t)), \tag{1}$$

$$\dot{y}_2(t) = -b(t)y_1(t-\tau(t)), \tag{2}$$

where $\tau, a, b\colon I_0 := [t_0, \infty) \to R^+ = (0, \infty)$, $t_0 \in R$, the difference $t - \tau(t)$ is nondecreasing, and a, b are continuous functions. The symbol " $\dot{}$ " stands for the *right-hand* derivative. Similarly, if necessary, the value of a function at a point is understood as the value of the corresponding limit *from the right*.

We deal with the existence of a positive solution of system (1), (2) for $t \to \infty$. The term *positive solution*, in the case of a system, stands for a solution having positive coordinates on an interval considered. Similarly we use the term *strictly decreasing* solution. Existence of positive solutions of system (1), (2) is connected with the existence of a positive solution of the scalar linear homogeneous differential equations with time lag

$$\dot{w}(t) = -\sqrt{a(t)b(t)}\, w(t-\tau(t)). \tag{3}$$

Theorem 1 *For existence of a positive and strictly decreasing solution $w = w(t)$ of (3) on $[t_0 - \tau(t_0), \infty)$ is necessary and sufficient existence of a locally integrable vector $\lambda\colon [t_0 - \tau(t_0), \infty) \to (0, \infty)$, continuous on $[\, t_0 - \tau(t_0), t_0) \cup I_0$, and satisfying the integral inequality*

$$\lambda(t) \ge \sqrt{a(t)b(t)}\, \exp\left[\int_{t-\tau(t)}^{t} \lambda(s)ds\right],$$

for $t \ge t_0$.

Theorem 2 *For existence of a positive and strictly decreasing solution $y = y(t) = (y_1(t), y_2(t))$ of the system (1), (2) on $[t_0 - \tau(t_0), \infty)$ is necessary and sufficient existence of a positive constant vector $k = (k_1, k_2)$ and a locally integrable vector $\lambda = (\lambda^1, \lambda^2)\colon [t_0 - \tau(t_0), \infty) \to (0, \infty) \times (0, \infty)$, continuous on $[\, t_0 - \tau(t_0), t_0) \cup I_0$, satisfying the system of integral inequalities*

$$\lambda^1(t) \ge \frac{k_2}{k_1} \cdot a(t) \cdot \exp\left[-\int_{t_0-\tau(t_0)}^{t-\tau(t)} \lambda^2(s)ds\right] \cdot \exp\left[\int_{t_0-\tau(t_0)}^{t} \lambda^1(s)ds\right],$$

$$\lambda^2(t) \ge \frac{k_1}{k_2} \cdot b(t) \cdot \exp\left[-\int_{t_0-\tau(t_0)}^{t-\tau(t)} \lambda^1(s)ds\right] \cdot \exp\left[\int_{t_0-\tau(t_0)}^{t} \lambda^2(s)ds\right]$$

for $t \geq t_0$.

Theorem 3 *If system* (1), (2) *has a positive and strictly decreasing solution on interval* $[t_0 - \tau(t_0), \infty)$ *then equation* (3) *has a positive and strictly decreasing solution on interval* $[t_0 - \tau(t_0), \infty)$.

Our effort in the future will be focused on the question if the existence of positive solutions of system (1), (2) is equivalent (in a sense) with the existence of positive solutions of equation (3). We formulate:

Conjecture 1 *The following two statements are equivalent:*

a) *System* (1), (2) *has a positive and strictly decreasing solution on interval* $[t_0 - \tau(t_0), \infty)$.

b) *Equation* (3) *has a positive and strictly decreasing solution on interval* $[t_0 - \tau(t_0), \infty)$.

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References

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