Asymptotic properties of solutions of the discrete analogue of the Emden-Fowler equation

Josef Diblík
Brno University of Technology, Faculty of Civil Engineering,
Department of Mathematics and Descriptive Geometry
e-mail: diblik.j@fce.vutbr.cz

Irena Hlavičková
Brno University of Technology, Faculty of Electrical Engineering and Communication,
Department of Mathematics
e-mail: hlavicka@feec.vutbr.cz

This contribution is devoted to the investigation of the asymptotic behavior of the solutions of the second-order difference equation

$$\Delta^2 v(k) + \frac{(\alpha-1)(\Delta v(k))^2}{v(k)} - \Delta v(k) + 1 = 0, \tag{1}$$

where $\alpha \in \mathbf{R}$, $\alpha < 0$, $k \in N(a) := \{a, a+1, \dots\}$, $a \in \mathbf{N}$, and $\Delta v(k) = v(k+1) - v(k)$.
This equation is the discrete analogue of the Emden-Fowler differential equation.
Equation (1) can be rewritten as a system of two first-order difference equations

$$\begin{aligned} \Delta u_1(k) &= u_1(k) - \frac{\alpha-1}{k(1+u_2(k))} \cdot [1+u_1(k)]^2, \\ \Delta u_2(k) &= \frac{1}{k+1}[-u_2(k) + u_1(k)], \end{aligned} \tag{2}$$

where $v(k) = k(1+u_2(k))$ and $\Delta v(k) = 1 + u_1(k)$.
System (2) is a special case of the general system of two difference equations

$$\begin{aligned} \Delta u_1(k) &= f_1(k, u_1(k), u_2(k)), \\ \Delta u_2(k) &= f_2(k, u_1(k), u_2(k)) \end{aligned} \tag{3}$$

with $f_1, f_2 : N(a) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$.
In one of the previous papers of the authors (see [1]) one can find sufficient conditions guaranteeing the existence of at least one solution $u(k) = (u_1^*(k), u_2^*(k))$, $k \in N(a)$, of system (3) satisfying

$$\begin{aligned} b_1(k) &< u_1^*(k) < c_1(k), \\ b_2(k) &< u_2^*(k) < c_2(k) \end{aligned}$$

where $b_i, c_i : N(a) \to \mathbf{R}$, $i = 1, 2$, are auxiliary functions such that $b_i(k) < c_i(k)$ for every $k \in N(a)$.
Applying this general result to equation (2), in the same paper it was shown that there exists a solution of system (2) satisfying for k sufficiently large the conditions

$$-\left(\frac{1}{k}\right)^{\nu_i} < u_i(k) < \left(\frac{1}{k}\right)^{\nu_i} \tag{4}$$

for $i = 1, 2$, where $0 < \nu_2 < \nu_1 < 1$. Rewritten in the terms of the second order equation (1) , it gives

$$|v(k) - k| < k \cdot \left(\frac{1}{k}\right)^{\nu_2}$$

and

$$|\Delta v(k) - 1| < \left(\frac{1}{k}\right)^{\nu_1} .$$

When this result was presented, the question came from the audience, whether this estimate of the solution could not be improved. Our present contribution gives a partial answer to this question (the investigation of this problem continues).

Theorem 1 *Let numbers ν_1, ν_2, $1 < \nu_1 < 2$, $0 < \nu_2 < 1$, $1+\nu_2 > \nu_1$, be given. Then the system of difference equations (2) has for sufficiently large $a \in \mathbf{N}$ a solution $u(k) = (u_1(k), u_2(k))$ such that*

$$\frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} < u_1(k) < \frac{\alpha - 1}{k} + \left(\frac{1}{k}\right)^{\nu_1},$$
$$-\left(\frac{1}{k}\right)^{\nu_2} < u_2(k) < \left(\frac{1}{k}\right)^{\nu_2}$$

for $k \in N(a)$. In other words, there exists a solution $v(k)$ of equation (1), such that

$$|v(k) - k| < k \cdot \left(\frac{1}{k}\right)^{\nu_2}$$

and

$$\left|\Delta v(k) - 1 - \frac{\alpha - 1}{k}\right| < \left(\frac{1}{k}\right)^{\nu_1}$$

for k sufficiently large.

Acknowledgment

This investigation was supported by the Council of Czech Government MSM 00216 30503 and by the grant No 1/3238/06 of the Grant Agency of Slovak Republic (VEGA).

References

[1] J. Diblík, I. Růžičková: *Compulsory asymptotic behavior of solutions of two-dimensional systems of difference equations,* Difference Equations and Discrete Dynamical Systems, Proceedings of the 9th International Conference, University of Southern California, Los Angeles, California, USA, 2–7 August 2004. World Scientific Publishing Co. Pte. Ltd, Singapore, 2005, 35–49.