Asymptotic properties of solutions of the discrete analogue of the Emden-Fowler equation

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Abstract

This contribution is devoted to the investigation of the asymptotic behavior of the solution of a special type of second-order difference equation. The considered equation is the discrete analogue of the Emden-Fowler differential equation. In paper [9] the authors have already found some asymptotic estimates of its solution. The aim of the presented paper is to improve these estimates.

1 Introduction

We will investigate the asymptotic behavior of the solutions of the second-order difference equation

$$\Delta^2 v(k) + \frac{(\alpha-1)(\Delta v(k))^2}{v(k)} - \Delta v(k) + 1 = 0, \tag{1}$$

where $\alpha \in \mathbf{R}$, $\alpha < 0$, $k \in N(a) := \{a, a+1, \dots\}$, $a \in \mathbf{N}$, and $\Delta v(k) = v(k+1) - v(k)$.
This equation is the discrete analogue of the Emden-Fowler differential equation. (The well-known Emden-Fowler equation is investigated e.g. in the book [2, Chapter VII].)
Equation (1) can be rewritten as a system of two first-order difference equations

$$\begin{aligned} \Delta u_1(k) &= u_1(k) - \frac{\alpha-1}{k(1+u_2(k))} \cdot (1+u_1(k))^2, \\ \Delta u_2(k) &= \frac{1}{k+1}(-u_2(k) + u_1(k)), \end{aligned} \tag{2}$$

where $v(k) = k(1+u_2(k))$ and $\Delta v(k) = 1 + u_1(k)$.
System (2) is a special case of the general system of two difference equations

$$\begin{aligned} \Delta u_1(k) &= f_1(k, u_1(k), u_2(k)), \\ \Delta u_2(k) &= f_2(k, u_1(k), u_2(k)) \end{aligned} \tag{3}$$

with $f_1, f_2 : N(a) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$.
In paper [9] one can find sufficient conditions guaranteeing the existence of at least one solution $u(k) = (u_1^*(k), u_2^*(k))$, $k \in N(a)$, of system (3) satisfying

$$\begin{aligned} b_1(k) &< u_1^*(k) < c_1(k), \\ b_2(k) &< u_2^*(k) < c_2(k) \end{aligned}$$

where $b_i, c_i : N(a) \to \mathbf{R}$, $i = 1, 2$, are auxiliary functions such that $b_i(k) < c_i(k)$ for every $k \in N(a)$.
The main result of [9] can be summarized in the following theorem.

Theorem 1 *Let $b_i(k), c_i(k), b_i(k) < c_i(k), i = 1, 2$, be real functions defined on $N(a)$ and let $f_i : N(a) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}, i = 1, 2$ be functions that are continuous with respect to their last two arguments.*
Suppose that for all points (u_1, u_2), such that $b_1(k) \le u_1 \le c_1(k)$ and $b_2(k) \le u_2 \le c_2(k)$ for some $k \in N(a)$, the following four conditions hold:

$$u_1 = b_1(k) \;\Rightarrow\; f_1(k, u_1, u_2) < b_1(k+1) - b_1(k), \tag{4}$$
$$u_1 = c_1(k) \;\Rightarrow\; f_1(k, u_1, u_2) > c_1(k+1) - c_1(k), \tag{5}$$
$$u_2 = b_2(k) \;\Rightarrow\; b_2(k+1) - b_2(k) < f_2(k, u_1, u_2) < c_2(k+1) - b_2(k), \tag{6}$$
$$u_2 = c_2(k) \;\Rightarrow\; b_2(k+1) - c_2(k) < f_2(k, u_1, u_2) < c_2(k+1) - c_2(k). \tag{7}$$

Let, moreover, function $F(w) = w + f_2(k, u_1, w)$ be monotone for every fixed $(k, u_1) \in \{(k, u_1) : k \in N(a), b_1(k) \le u_1 \le c_1(k)\}$ on the interval $b_2(k) \le w \le c_2(k)$.
Then there exists a solution $u = (u_1^*(k), u_2^*(k))$ of system (3) satisfying the inequalities

$$b_1(k) < u_1^*(k) < c_1(k),$$
$$b_2(k) < u_2^*(k) < c_2(k)$$

for every $k \in N(a)$.

Applying this general result to equation (2), in the same paper it was shown that there exists a solution of system (2) satisfying for k sufficiently large the conditions

$$-\left(\frac{1}{k}\right)^{\nu_i} < u_i(k) < \left(\frac{1}{k}\right)^{\nu_i} \tag{8}$$

for $i = 1, 2$, where $0 < \nu_2 < \nu_1 < 1$. Rewritten in the terms of the second order equation (1) , it gives

$$|v(k) - k| < k \cdot \left(\frac{1}{k}\right)^{\nu_2}$$

and

$$|\Delta v(k) - 1| < \left(\frac{1}{k}\right)^{\nu_1} .$$

When this result was presented, the question came from the audience, whether this estimate of the solution could not be improved. Our present contribution gives a partial answer to this question (the investigation of this problem continues).

2 New asymptotic estimate of the solution

Theorem 2 *Let numbers ν_1, ν_2, $1 < \nu_1 < 2$, $0 < \nu_2 < 1$, $1+\nu_2 > \nu_1$, be given. Then the system of difference equations (2) has for sufficiently large $a \in \mathbf{N}$ a solution $u(k) = (u_1(k), u_2(k))$ such that*

$$\frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} < u_1(k) < \frac{\alpha - 1}{k} + \left(\frac{1}{k}\right)^{\nu_1}, \tag{9}$$

$$-\left(\frac{1}{k}\right)^{\nu_2} < u_2(k) < \left(\frac{1}{k}\right)^{\nu_2} \tag{10}$$

for $k \in N(a)$. In other words, there exists a solution $v(k)$ of equation (1), such that

$$|v(k) - k| < k \cdot \left(\frac{1}{k}\right)^{\nu_2}$$

and

$$\left|\Delta v(k) - 1 - \frac{\alpha - 1}{k}\right| < \left(\frac{1}{k}\right)^{\nu_1}$$

for k sufficiently large.

Proof. Verify the conditions of Theorem 1 with

$$\begin{aligned} f_1(k, u_1, u_2) &= u_1 - \frac{\alpha - 1}{k(1 + u_2)}(1 + u_1)^2, \\ f_2(k, u_1, u_2) &= \frac{1}{k+1}(-u_2 + u_1), \end{aligned}$$

$$b_1(k) = \frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1}, \qquad c_1(k) = \frac{\alpha - 1}{k} + \left(\frac{1}{k}\right)^{\nu_1},$$

$$b_2(k) = -\left(\frac{1}{k}\right)^{\nu_2}, \qquad c_2(k) = \left(\frac{1}{k}\right)^{\nu_2}.$$

With respect to condition (4) we have to verify that

$$f_1(k, u_1, u_2) < b_1(k+1) - b_1(k)$$

if

$$u_1 = \frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \text{ and } -\left(\frac{1}{k}\right)^{\nu_2} \le u_2 \le \left(\frac{1}{k}\right)^{\nu_2},$$

which gives

$$\frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha - 1}{k(1+u_2)}\left(1 + \frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1}\right)^2 < \frac{\alpha - 1}{k+1} - \left(\frac{1}{k+1}\right)^{\nu_1} - \frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \tag{11}$$

Denote $L_{(11)}$ and $R_{(11)}$ the left-hand and the right-hand side of inequality (11), respectively. In the following estimate of $L_{(11)}$ we will use the fact that $\alpha < 0$, i.e. also $\alpha - 1 < 0$, and the fact that $\frac{1}{1-x} < 1 + 2x$ for $x > 0$ sufficiently close to zero.

$$\begin{aligned} L_{(11)} &\le \frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha - 1}{k(1+u_2)}\left(1 + \frac{\alpha - 1}{k}\right)^2 \le \\ &\le \frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha - 1}{k(1 - (\frac{1}{k})^{\nu_2})}\left(1 + \frac{\alpha - 1}{k}\right)^2 \le \\ &\le \frac{\alpha - 1}{k} - \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha - 1}{k}\left(1 + 2\left(\frac{1}{k}\right)^{\nu_2}\right)\left(1 + 2\frac{\alpha - 1}{k} + \frac{(\alpha - 1)^2}{k^2}\right) = \\ &= \frac{\alpha - 1}{k} - \frac{1}{k^{\nu_1}} - \frac{\alpha - 1}{k}\left(1 + 2\frac{\alpha - 1}{k} + \frac{(\alpha - 1)^2}{k^2} + 2\frac{1}{k^{\nu_2}} + 4\frac{\alpha - 1}{k^{1+\nu_2}} + 2\frac{(\alpha - 1)^2}{k^{2+\nu_2}}\right) = \\ &= -\frac{1}{k^{\nu_1}} - 2\frac{(\alpha - 1)^2}{k^2} - \frac{(\alpha - 1)^3}{k^3} - 2\frac{\alpha - 1}{k^{1+\nu_2}} - 4\frac{(\alpha - 1)^2}{k^{2+\nu_2}} - 2\frac{(\alpha - 1)^3}{k^{3+\nu_2}} \end{aligned}$$

As, due to the assumptions of the Theorem, $1 < \nu_1 < 2$, $0 < \nu_2 < 1$ and $1 + \nu_2 > \nu_1$, one can state that $L_{(11)}$ is negative for k sufficiently large.

As for $R_{(11)}$, it can be simplified to

$$R_{(11)} = -\frac{\alpha-1}{k(k+1)} + \left(\frac{1}{k}\right)^{\nu_1} - \left(\frac{1}{k+1}\right)^{\nu_1} > 0$$

Thus, inequality (11) holds.
Now let us prove inequality (5), i.e.

$$f_1(k, u_1, u_2) > c_1(k+1) - c_1(k)$$

if

$$u_1 = \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1} \text{ and } -\left(\frac{1}{k}\right)^{\nu_2} \le u_2 \le \left(\frac{1}{k}\right)^{\nu_2},$$

which gives

$$\frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha-1}{k(1+u_2)}\left(1+\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}\right)^2 > \frac{\alpha-1}{k+1} + \left(\frac{1}{k+1}\right)^{\nu_1} - \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \tag{12}$$

Again, denote $L_{(12)}$ and $R_{(12)}$ the left-hand and the right-hand side of (12), respectively. This time we will use the fact that $\frac{1}{1+x} > 1-2x$ for $x > 0$.

$$\begin{aligned}
L_{(12)} &\ge \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha-1}{k(1+u_2)}\left(1+\frac{\alpha-1}{k}\right)^2 \ge \\
&\ge \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha-1}{k(1+(\frac{1}{k})^{\nu_2})}\left(1+\frac{\alpha-1}{k}\right)^2 \ge \\
&\ge \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha-1}{k}\left(1-2\left(\frac{1}{k}\right)^{\nu_2}\right)\left(1+2\,\frac{\alpha-1}{k}+\frac{(\alpha-1)^2}{k^2}\right) = \\
&= \frac{\alpha-1}{k} + \frac{1}{k^{\nu_1}} - \frac{\alpha-1}{k}\left(1+2\,\frac{\alpha-1}{k}+\frac{(\alpha-1)^2}{k^2} - 2\,\frac{1}{k^{\nu_2}} - 4\,\frac{\alpha-1}{k^{1+\nu_2}} - 2\,\frac{(\alpha-1)^2}{k^{2+\nu_2}}\right) = \\
&= \frac{1}{k^{\nu_1}} - 2\,\frac{(\alpha-1)^2}{k^2} - \frac{(\alpha-1)^3}{k^3} + 2\,\frac{\alpha-1}{k^{1+\nu_2}} + 4\,\frac{(\alpha-1)^2}{k^{2+\nu_2}} + 2\,\frac{(\alpha-1)^3}{k^{3+\nu_2}} = O\left(\frac{1}{k^{\nu_1}}\right)
\end{aligned}$$

For the estimate of $R_{(12)}$, let us use the fact (gained with the help of the Mean Value Theorem) that

$$\left(\frac{1}{k+1}\right)^{\nu_1} - \left(\frac{1}{k}\right)^{\nu_1} = -\nu_1\left(\frac{1}{\xi}\right)^{\nu_1+1}, \tag{13}$$

where $k \le \xi \le k+1$. That gives

$$R_{(12)} \le -\frac{\alpha-1}{k(k+1)} - \frac{\nu_1}{(k+1)^{1+\nu_1}} = O\left(\frac{1}{k^2}\right)$$

and inequality (12) is verified.
The proofs of inequalities (6) and (7) is easier.
First let us take the left part of inequality (6):

$$b_2(k+1) - b_2(k) < f_2(k, u_1, u_2)$$

if

$$u_2 = -\left(\frac{1}{k}\right)^{\nu_2} \text{ and } \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \le u_1 \le \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1},$$

which gives

$$-\left(\frac{1}{k+1}\right)^{\nu_2}+\left(\frac{1}{k}\right)^{\nu_2}<\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}+u_1\right) \tag{14}$$

Using a relation similar to (13), the left-hand side of (14) can be estimated as follows

$$-\left(\frac{1}{k+1}\right)^{\nu_2}+\left(\frac{1}{k}\right)^{\nu_2}\leq\frac{\nu_2}{k^{1+\nu_2}}$$

The right-hand side of (14) can be estimated as

$$\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}+u_1\right)\geq\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}+\frac{\alpha-1}{k}-\left(\frac{1}{k}\right)^{\nu_1}\right)=\frac{1}{(k+1)k^{\nu_2}}+O\left(\frac{1}{k^2}\right)$$

That proves inequality (14) because $\nu_2<1$ and thus

$$\frac{\nu_2}{k^{1+\nu_2}}<\frac{1}{(k+1)k^{\nu_2}}+O\left(\frac{1}{k^2}\right)$$

for k sufficiently large.
The right part of inequality (6), i.e.

$$f_2(k,u_1,u_2)<c_2(k+1)-b_2(k)$$

if

$$u_2=-\left(\frac{1}{k}\right)^{\nu_2}\text{ and }\frac{\alpha-1}{k}-\left(\frac{1}{k}\right)^{\nu_1}\leq u_1\leq\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1},$$

in our case gives

$$\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}+u_1\right)<\left(\frac{1}{k+1}\right)^{\nu_2}+\left(\frac{1}{k}\right)^{\nu_2}. \tag{15}$$

The maximum possible value of the left-hand side of (15) is

$$\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}+\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}\right)=O\left(\frac{1}{k^{1+\nu_2}}\right),$$

meanwhile the right-hand side of (15) is $O\left(\frac{1}{k^{\nu_2}}\right)$ and hence inequality (15) holds.
Condition (7) can be proved in a very similar way. First its left part:

$$b_2(k+1)-c_2(k)<f_2(k,u_1,u_2)$$

if

$$u_2=\left(\frac{1}{k}\right)^{\nu_2}\text{ and }\frac{\alpha-1}{k}-\left(\frac{1}{k}\right)^{\nu_1}\leq u_1\leq\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}.$$

That gives

$$-\left(\frac{1}{k+1}\right)^{\nu_2}-\left(\frac{1}{k}\right)^{\nu_2}<\frac{1}{k+1}\left(-\left(\frac{1}{k}\right)^{\nu_2}+u_1\right).$$

Multiplying it by -1, we get

$$\left(\frac{1}{k+1}\right)^{\nu_2}+\left(\frac{1}{k}\right)^{\nu_2}>\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}-u_1\right). \tag{16}$$

As the left-hand side of (16) is $O\left(\frac{1}{k^{\nu_2}}\right)$ and the right-hand side is $O\left(\frac{1}{k^{1+\nu_2}}\right)$, inequality (16) has be fulfilled.

Now prove the right part of condition (7):

$$f_2(k,u_1,u_2) < c_2(k+1) - c_2(k)$$

if

$$u_2 = \left(\frac{1}{k}\right)^{\nu_2} \text{ and } \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \le u_1 \le \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1},$$

i.e.

$$\frac{1}{k+1}\left(-\left(\frac{1}{k}\right)^{\nu_2} + u_1\right) < \left(\frac{1}{k+1}\right)^{\nu_2} - \left(\frac{1}{k}\right)^{\nu_2}.$$

Again we will change the signes:

$$\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2} - u_1\right) > \left(\frac{1}{k}\right)^{\nu_2} - \left(\frac{1}{k+1}\right)^{\nu_2}. \tag{17}$$

The left-hand side of (17) can be expressed as

$$\frac{1}{(k+1)k^{\nu_2}} + O\left(\frac{1}{k^2}\right),$$

the maximum possible value of the right hand side is $\nu_2/k^{1+\nu_2}$ and thus inequality (17) holds. Finally, the function

$$F(w) = w + f_2(k,u_1,w) = w + \frac{1}{k+1}(-w+u_1)$$

is monotone for every fixed (k,u_1) such that $k \in N(a), b_1(k) \le u_1 \le c_1(k)$ on the interval $b_2(k) \le w \le c_2(k)$ since its derivative

$$F'(w) = 1 - \frac{1}{k+1} = \frac{k}{k+1}$$

remains (for sufficiently large $a \in \mathbf{N}$) always positive. Then, in accordance with the statement of Theorem 1, there exists a solution of system (2) satisfying inequalities (9) and (10).

Acknowledgment

This investigation was supported by the Council of Czech Government MSM 00216 30503 and by the grant No 1/3238/06 of the Grant Agency of Slovak Republic (VEGA).

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