When certain basic hyperoperations on lattices coincide

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Investigation of algebraic structures particularly in noncommutative algebra leads in natural way to hyperstructures formerly called multistructures - [3–5, 7, 20, 23]. More in detail, decomposition of non-commutative groups by their non-invariant subgroups, or decomposition of rings by their subrings which are not ideals allow to creat structures with many-valued operations. Other motivation is coming from geometry. Analysis of geometrical structures leads to various binary hyperstructures, in particular, to the notion of a join space which has been introduces by Walter Premowitz and used by him and afterwards together with James Jantosciak to build again several branches of geometry. Moreover, further motivation for investigation and application of hyperstructures can be found in chemistry and in nuclear physics.

Hyperstructures formed by various operators (transformation operators of real or complex functions, differential or integral operators) show that constructions with input hyperstructures on centralizers of elements of suitable semigroups (groups) are of a certain interest.

Recall that the present actual aim of the mathematical education of students on technical universities and classical universities, as well, is the transfer of new actual scientific results into curricula according to an appropriate didactical system of mathematical education.

Now recall some basic notions and notation of the hypergroup theory – for further information cf. [3, 4, 7]. A *hypergrupoid* is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet : H \times H \to \mathcal{P}^*(H)$ is a binary hyperoperation on H. Symbol $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of H. If the associativity axiom $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a, b, c \in H$, then the pair $(H, \bullet)$ is called a *semihypergroup*. If moreover the reproduction axiom $a \bullet H = H = H \bullet a$ is satisfied for any element $a \in H$, then the pair $(H, \bullet)$ is called a *hypergroup*. A hypergrupoid $(H, \bullet)$, where the reproduction axiom holds, is called a *quasi-hypergroup*.

We can define a *hyperproduct* of any pair of nonempty subsets of $A, B \in H$ as $A \bullet B = \cup\{a \bullet b\ a \in A, b \in B\}$.

A hypergroup $(H, \bullet)$ is called a *transposition hypergroup* or a *join space* if it satisfies the following transposition axiom: For all $a, b, c, d \in H$ the relation $b\backslash a \approx c/d$ implies $a \bullet d \approx b \bullet c$, where $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$. Sets $b\backslash a = \{x \in H; a \in b \bullet x\}$ and $c/d = \{x \in H; c \in x \bullet d\}$ are called *left* and *right extensions*, or *fractions*, respectively

We describe a simple but important construction from [7], which has been used also in [8] or [12] and elsewhere, which enables us to obtain in a certain sense analogous results to those presented in this contribution.

By a *quasi-ordered semigroup* we mean a triple $(G, \bullet, \leq)$, where $(G, \bullet)$ is a semigroup and binary relation $\leq$ is a quasi-ordering (i.e. it is reflexive and transitive) on the set G such, that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \bullet z \leq y \bullet z$ and $z \bullet x \leq z \bullet y$ hold. By an *ordered* (*semi*)*group* we mean (as usual) a triple $(G, \bullet, \leq)$, where $(G, \bullet)$ is a (semi)group and $\leq$ is a reflexive, antisymmetrical and transitive binary relation on G such, that for any triple $x, y, z \in G$ with the property $x \le y$ also $x \bullet z \le y \bullet z$ and $z \bullet x \le z \bullet y$ hold. Further, $[a)_{\le} = \{x \in G; a \le x\}$ is a principal end generated by $a \in G$. To any element a of a noncommutative group there is assigned a pair of mappings $\lambda_a : G \to G$, $\rho_a : G \to G$ defined by $\lambda_a(x) = a \bullet x$, $\rho_a(x) = x \bullet a$. These are called *left* and *right translation determined by* $a \in G$ respectively. Notice that a group with an ordering $(G, \bullet, \le)$ is an ordered group if and only if all its left and right translations $\lambda_a, \rho_a, a \in S$, are order-preserving, i.e. isotone selfmaps of the ordered set $(S, \le)$. By an *inclusion homomorphism* we mean a mapping $f : (G, \bullet_G) \to (H, \bullet_H)$ such that $f(a \bullet_H b) \subset f(a) \bullet_G f(b)$.

The following lemma, which is crucial for our considerations, is proved in [12]; for the first time in [7], pp 146–7.

Lemma. *Let a triple $(G, \cdot, \le)$ be a quasi-ordered semigroup. Define a hyperoperation*

$$* : G \times G \to \mathcal{P}^*(G) \qquad \text{by} \qquad a * b = [a \cdot b)_{\le} = \{x \in G; a \cdot b \le x\}$$

for all pairs of elements $a, b \in G$.

1. *Then $(G, *)$ is a semihypergroup which is commutative if the semigroup $(G, \cdot)$ is commutative.*
2. *Let $(G, *)$ be the above defined semihypergroup. Then $(G, *)$ is a hypergroup iff for any pair of elements $a, b \in G$ there exists a pair of elements $c, c' \in G$ with a property $a \cdot c \le b, c' \cdot a \le b$.*

Remark. *Notice that if $(G, \cdot, \le)$ is a quasi-ordered group than the condition stated under 2. is satisfied, hence the final hyperstructure is a hypergroup.*

Example. Notice that this example is a certain simplification of that presented in [15], example 3. So, suppose $I \subset \mathbb{R}$ is an open interval of the set $\mathbb{R}$ of all real numbers. We denote by$\mathbb{L}d_2(I)$ the set of all operators $L(p, q) : \mathbb{C}^2(I) \to \mathbb{C}(I)$ of the form $L(p, q) = \frac{d^2}{dx^2} + p\frac{d}{dx} + qId$, when $p = p(x)$, $q = q(x)$ are continuous functions. Thus, for any function $y \in \mathbb{C}^2(I)$ (the set of all functions with continuous second derivatives) we have $L(p, q)y = y'' + py' + qy$. For any pair of differential operators $L(p_1, q_1)$, $L(p_2, q_2) \in \mathbb{L}d_2(I) = \big\{L(p, q) : p, q \in \mathbb{R}, \ p > 0\big\}$ we define

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1 p_2, p_1 q_2 + q_1)$$

and $L(p_1, q_1) \le L(p_2, q_2)$ whenever $p_1 = p_2$, $q_1 \le q_2$. By Proposition 1, [8] page 79 we have that $(\mathbb{L}d_2(I), \cdot, \le)$ is a noncommutative ordered group with the unit element $L(1, 0)$. The proof of the mentioned proposition is contained in pages 79, 80. The set $\mathbb{L}d_2(I)$ can be endowed by structure of a noncommutative hypergroups it the same way as in paper [10]. More precisely setting

$$\begin{aligned} L(p_1, q_1) * L(P_2, q_2) &= \big\{L(p, q) \in \mathbb{L}d_2(I) : L(p_1, q_1) \cdot L(p_2, q_2) \le L(p, q)\big\} \\ &= \big\{(p_1, p_2, q) : q \in \mathbb{C}(I), p_1 q_2 + q_1 \le q\big\} \end{aligned}$$

for arbitrary pairs of operators $L(p_1, q_1)$, $L(p_2, q_2) \in \mathbb{L}d_2(I)$, we obtain that the hypergroupoid $\big(\mathbb{L}d_2(I), *\big)$ is a (noncommutative) hypergroup.

The above "ending lemma" allows to realize constructions of hyperstructures based on quasi-ordered or ordered groups or semigroups of various translation operators, affine transformations, ordinary differential operators, partial differential operators or integral operators extracted from Fredholm integral equations of the first and the second kind. This is done e.g. in papers [1, 2, 6, 8, 10, 11, 12, 14, 15, 17, 18, 23].

The further useful construction is the assignement of a commutative hypergroup (in the sense of Marty) to an ordered set. This idea can be found in several papers. So, if $(S, \leqq)$ is an ordered set then putting

$$\gamma(a, b) = \{x \in S; a \leqq x \text{ or } b \leqq x\}$$

we obtain that (S,γ) is a commutative hypergroup (cf. [7, 23]. Moreover, putting dually

$$\delta(a,b)=\{x\in S; x\leqq a \text{ or } x\leqq b\}$$

then we get two functorial transfers from the category of ordered sets and isotone mappings into the category of commutative hypergroups and their homomorphisms. It has been obtained that these functors are realizations of the first category in the second one and that there exist exactly these two mentioned realizations up to natural isomorphism. The main theorem of this contribution show the situation when some of the above nmentioned constructions yield the same result.

In connection with the above remarks we are going to observe relationships between certain basic hyperoperations on lattices.

As usually by a lattice we mean a set L endowed with two binary operations $\vee$ (join) and $\wedge$ (meet), i.e. mappings $\vee : L\times L\to L$, $\wedge : L\times L\to L$ satisfying these axioms:

For all elements a, b, $c\in L$ there holds

(i) $a\vee a=a$, $a\wedge a=a$ (the idempotency law),

(ii) $a\vee b=b\vee a$, $a\wedge b=b\wedge a$ (the commutativity law),

(iii) $(a\vee b)\vee c=a\vee(b\vee c)$, $(a\wedge b)\wedge c=a\wedge(b\wedge c)$ (the associativity law),

(iv) $a\vee(b\wedge c)=a$, $a\wedge(b\vee a)=a$ (the absorption law).

Further, for $a,b\in L$ we define $a\leq b$ whenever these equivalent equalities are valid:

$$a\wedge b=a\,,\quad a\vee b=b\,.$$

So, any lattice $(L,\vee,\wedge)$ is the ordered set $(L,\leq)$. For an element $a\in L$ we denote by

$$[a)_{\leq}=\{x\in L; a\leq x\}\,,\quad \big((a]_{\leq}=\{x\in L; x\leq a\}\big)$$

the principal end or the principal upper cone (the principal beginning or the principal lower cone) generated by the element $a\in L$. Now we define binary hyperoperations $\varkappa_i : L\times L\to\mathcal{P}^*(L)$ (the power set of L without the empty subset $\emptyset$), $i=1,2$,

$$\lambda_i : L\times L\to\mathcal{P}^*(L)\,,\quad \varrho_i : L\times L\to\mathcal{P}^*(L)\,,\qquad i=1,2$$

and the interval hyperoperation $\sigma : L\times L\to\mathcal{P}^*(L)$ by these rules:

$$\begin{aligned}
\varkappa_1(a,b)&=[a)_{\leq}\cup[b)_{\leq}\,,\quad \varkappa_2(a,b)=(a]_{\leq}\cup(b]_{\leq}\,,\\
\lambda_1(a,b)&=[a\vee b)=[a)\cap[b)\,,\quad \lambda_2(a,b)=(a\vee b]\,,\\
\varrho_1(a,b)&=[a\wedge b)\,,\quad \varrho_2(a,b)=(a\wedge b]=(a]\cap(b]\,,\\
\sigma(a,b)&=[a\wedge b,a\vee b]=\{x\in L; a\wedge b\leq x\leq a\vee b\}
\end{aligned}$$

for any pair of elements $a,b\in L$.

In connection with fuzzy algebras and so called minimax algebras there are playing an important role bottleneck algebras.

A bottleneck algebra is defined to be a triple $(R,\oplus,\otimes)$, where R is linearly ordered set, $\oplus$ and $\otimes$ are binary operations on R such that for each $a,b\in R$ we have $a\oplus b=\max\{\text{a, b}\}$, $a\otimes b=\min\{a,b\}$ (cf. [19] p. 59).

Theorem. *Let* $(L, \vee, \wedge)$ *be a lattice,* $\varkappa_i$, λ_i, ϱ_i, $i = 1, 2$ *and* σ *be the above defined binary hyperoperations on the set* L. *The following conditions are equivalent:*

(1) $\varkappa_1 = \varrho_1$,

(2) $\varkappa_2 = \lambda_2$,

(3) *for any pair of elements* $a, b \in L$ *we have* $\lambda_1(a, b) \approx \{a, b\}$,

(4) *for any pair of elements* $a, b \in L$ *we have* $\varrho_2(a, b) \approx \{a, b\}$,

(5) *for any quadruple* a, b, c, $d \in L$ *such that* $a \neq b$, $c \neq d$ *the equality* $\sigma(a, b) = \sigma(c, d)$ *implies either* $a = c$, $b = d$ *or* $a = d$, $b = c$,

(6) *the ordered set* $(L, \leq)$ *is a chain, i.e. the lattice* $(L, \vee, \wedge)$ *is a bottleneck algebra.*

Proof. We are going to verify this cycle of implications (1)⇒(2)⇒(3)⇒(4)⇒(6)⇒(5)⇒(1).

(1)⇒(2): Suppose condition (1) is satisfied, i.e. $[a)_\leq \cup [b)_\leq = [a \wedge b)_\leq$ for any pair of elements a, $b \in L$. Then for an arbitrary pair a, $b \in L$ we have $a \wedge b \in [a)_\leq$ or $a \wedge b \in [b)_\leq$ hence $a \leq a \wedge b$ or $b \leq a \wedge b$. Since $a \wedge b \leq a$, $a \wedge b \leq b$ either $a = a \wedge b$ or $b = a \wedge b$, consequently either $a \leq b$ or $b \leq a$. Then either $a \vee b = b$ and

$$\varkappa_2(a, b) = (a]_\leq \cup (b]_\leq = (b]_\leq = (a \vee b]_\leq = \lambda_2(a, b)$$

or $a \vee b = a$ and we have

$$\varkappa_2(a, b) = (a]_\leq \cup (b]_\leq = (a]_\leq = (a \vee b]_\leq = \lambda_2(a, b)\,,$$

thus $\varkappa_2 = \lambda_2$, i.e. condition (2) is satisfied.

(2)⇒(3): Suppose (2) holds and a, $b \in L$ is an abitrary pair of elements. Then $(a]_\leq \cup (b]_\leq = (a \vee b]_\leq$. Similarly as above either $a \vee b \leq a$ or $a \vee b \leq b$, i.e. either $a \vee b = a$ which is followed by $a \in [a \vee b)_\leq = \lambda_1(a, b)$, or $a \vee b = b$, which means $b \in [a \vee b)_\leq = \lambda_1(a, b)$, thus $\lambda_1(a, b) \approx \{a, b\}$, i.e. condition (3) is satisfied.

(3)⇒(4): Suppose (3) is satisfied. Let a, $b \in L$ be an arbitrary pair of elements. If $a \in \lambda_1(a, b) = [a \vee b)_\leq$, we have $a \vee b \leq a$, i.e. $a \vee b = a$, consequently $b \leq a$, i.e. $a \wedge b = b$, thus $b \in (a \wedge b]_\leq = \varrho_2(a, b)$. If $b \in \lambda_1(a, b)$ then similarly $a \wedge b = b$, i.e. $a \leq b$ which implies $a \in (a \wedge b]_\leq = \varrho_2(a, b)$. Consequently $\varrho_2(a, b) \approx \{a, b\}$, therefore condition (4) is satisfied.

(4)⇒(6): Suppose (4) holds. Let a, $b \in L$ be arbitrary elements. If $a \in (a \wedge b]_\leq$, i.e. $a \leq a \wedge b$ then $a = a \wedge b$, thus $a \leq b$. If $b \in (a \wedge b]_\leq$ then similarly $b = a \wedge b$, thus $b \leq a$. Consequently any two elements of the lattice $(L, \vee, \wedge)$ are comparable, therefore $(L, \leq)$ is a chain – condition (6) is satisfied.

(6)⇒(5): Suppose the ordered set $(L, \leq)$ is a chain. Let a, b, c, $d \in L$ be a quadruple such that $a \neq b$, $c \neq d$. Suppose $a < b$, $c < d$ and $\sigma(a, b) = \sigma(c, d)$. Since $a \wedge b = a$, $a \vee b = b$, $c \wedge d = c$, $c \vee d = d$ we have $[a, b]_\leq = [c, d]_\leq$ which implies $a = c$, $b = d$. If $b < a$ and $c < d$ then $a = d$, $b = c$. So, in all possible cases either $a = c$, $b = d$ or $a = d$, $b = c$. Hence condition (5) is satisfied.

(5)⇒(1): Suppose condition (5) is satisfied. Evidently $[a)_\leq \subset [a \wedge b)_\leq$, $[b)_\leq \subset [a \wedge b)_\leq$ which implies $\varkappa_1(a, b) = [a)_\leq \cup [b)_\leq \subset [a \wedge b)_\leq = \varrho_1(a, b)$ for any pair of elements a, $b \in L$. Now, since

$$\sigma(a \wedge b, a \vee b) = [a \wedge b, a \vee b]_\leq = \sigma(a, b)\,,$$

we have by (5) that either $a = a \wedge b$, $b = a \vee b$ or $a \wedge b = b$, $a \vee b = a$. In the first case

$$[a \wedge b)_\leq = [a)_\leq \subset [a)_\leq \cup [b)_\leq\,,$$

in the second case

$$[a \wedge b)_\leq = [b)_\leq \subset [a)_\leq \cup [b)_\leq\,,$$

thus

$$\varkappa_1(a, b) = [a)_\leq \cup [b)_\leq = [a \wedge b)_\leq = \varrho_1(a, b)$$

for any pair a, $b \in L$. $\square$

Remark. Let $(S, \cdot)$ be a commutative band, i.e. $(S, \cdot)$ is a commutative semigroup each element of which is idempotent $(a \cdot a = a)$. The natural ordering on a commutative band $(S, \cdot)$ defined by $a, b \in S$, $a \leqq b$ whenever $a \cdot b = a$ determines structure of a lower semilattice (i.e. an ordered set in which any pair of its elements has infimum). Now, if $(S, \cdot)$ is a commutative band, then putting

$$a \circ b = \{x \in S; a \cdot x = a \text{ or } \ b \cdot x = b\}$$

and

$$a * b = \{y \in S; a \cdot y \cdot b = a \cdot b\}$$

we obtain that $(S, \circ)$ is a commutative hypergroup and $(S, *)$ is a commutative semihypergroup. From the above theorem there follows that the semihypergroup $(S, *)$ is a hypergroup identical with$(S, \circ)$ if and only if for any pair of elements $a, b \in S$ either $a \cdot b = a$ or $a \cdot b = b$. Moreover, the obtained hypergroup is a join space.

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