Are graduates aware of uncertainty in modeling?

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1 Introduction

Although courses of mathematical and computer modeling form a part of undergraduate curricula at technical universities, a question arises whether or not they really introduce students, at least at an elementary level, to the fundamental features of modeling.

Generally speaking, the way students learn to apply mathematics is oversimplified. In courses of mathematics, students concentrate on solving mathematical problems with crisp input data. Of course, this sort of purified problems is a substantial and natural part of the education in mathematics. However, students put the emphasis on obtaining results and they neglect checking the validity of the obtained results even if such checking is easier and much faster than the solving process itself. Take, for instance, an indefinite integral whose correctness can be directly checked by differentiation.

Perhaps, students consider this sort of correctness proofs too demanding because they (students) are rarely sufficiently skilled in differentiation to see it as a useful tool and, moreover, they probably do not fully understand that integration and differentiation are mutually inverse processes.

Examples of gross negligence are even more explicit in mathematical problems that have clear geometrical meaning. How many students are surprised that their calculation of the area or volume of a 2D or 3D domain leads to a negative value?

In the light of this lack of ability to use elementary reasoning or standard methods, we cannot expect that students might use advanced tools or witty methods to check the results they inferred.

In computational modeling courses, students learn how to interact with a computer and with a particular software package designed to solve engineering problems. An experienced lecturer might demonstrate at least some pitfalls of modeling, but the usual shortage of time does not allow for a deeper insight. As a result, many students identify mathematical modeling with choosing an appropriate predefined model from a list, typing reasonable input data, pushing a `solve` button, and producing beautiful color figures that depict output values. Again, students do not much care whether or not the calculated results are correct.

An alarming lesson can be taken from all the above observations. Students, in general, are not accustomed to critically check the steps they take when applying mathematics. They are not doubtful about input data, computational methods, and output data.

2 Verification and validation

Four notions form the cornerstones of modeling (see I. Babuška and J. T. Oden: Verification and validation in computational engineering and science: basic concepts, *Comput. Methods in Appl. Mech. Engrg.* 193, 2004, 4057–4066):

Mathematical model: A collection of mathematical constructions that provide abstractions of a physical event consistent with a scientific theory proposed to cover that event.

Computational model: The discretized version of a mathematical model that has been designed to be implemented on (or to be processed by) a computer or a computational device.

Verification: The process of determining if a computational model obtained by discretizing a mathematical model of a physical event and the code implementing the computational model can be used to represent the mathematical model of the event with sufficient accuracy.

Validation: The process of determining if a mathematical model of a physical event represents the actual physical event with sufficient accuracy.

Students should be taught to be critical analysts and modelers. They should be aware of both a hierarchy of mathematical models born from an analyzed problem and a hierarchy of computational approximations to such models. They should be aware of the possible sources of uncertainty, inaccuracy, and errors involved in modeling. They should also know basic techniques for acquiring the confidence of the computational model. It is clear, however, that time limitations prevent students from learning more than the elements of the above-mentioned skills.

A particular aspect of uncertainty in modeling deserves special attention: problems with uncertain input data. These are closer to situations modeled in engineering than models with crisp input data. Although stochastic approaches to uncertain inputs are popular and widely used (Monte Carlo simulation), we limit ourselves to deterministic methods.

3 Problems with uncertain data; fuzzy input data

Let $u(a)$ be a solution to an a-dependent and uniquely solvable state problem; a boundary value problem or a variational inequality, for instance. It is assumed that a belongs to $\mathcal{U}_{\mathrm{ad}}$, a set of admissible parameters. These can be scalars as various coefficients and material parameters, or functions as coefficient-function in differential equations.

In practice, a particular feature of $u(a)$ (a quantity of interest) is in focus, and this is expressed through a criterion-functional Φ evaluating $u(a)$ and, possibly, also a, that is, values $\Phi(a, u(a))$ are investigated, where $a \in \mathcal{U}_{\mathrm{ad}}$.

Analysts are interested in extremes of $\Phi(a, u(a))$ over $\mathcal{U}_{\mathrm{ad}}$. If this extremalization problem is not directly solvable, an approximation is necessary. The state problem can be discretized by the finite element method, for instance, to obtain an approximate solution u_h. If $\mathcal{U}_{\mathrm{ad}}$ consists of functions, these are approximated by a subset of an M-dimensional space of functions such as splines, for example. As a consequence, the extremalization of $\Phi(a_M, u_h(a_M))$ is performed over $\mathcal{U}_{\mathrm{ad}}^M$, a set identifiable with an M-dimensional subset of $\mathbb{R}^M$. The fully discretized problem is nothing else but a constrained optimization problem where the cost function coincides with the criterion functional assessing the approximate state solution. To cope with this sort of extremalization problems, standard tools for sensitivity analysis and optimization are sufficient.

At first glance, these problems might seem quite demanding of students' knowledge and skills. Although this is essentially true, it is possible to pose many problems accessible to undergraduates. Take simple truss and beam structures with uncertain material or geometrical parameters, for instance. Often, a monotone dependence of the criterion-functional value on an uncertain parameter appears, and, consequently, simplifies the search for an extremum.

It is not difficult to incorporate fuzziness into $\mathcal{U}_{\mathrm{ad}}$. It turns out that then a sequence of extremalization problems has to be solved to obtain the membership function of the criterion functional values, which is the ultimate goal in the modeling of a fuzzy quantity of interest.

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