Regularity and FEM Error Estimates of Viscous Incompressible Stokes Flow in Polygonal Domains near Corners

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Let Ω be a polygonal domain in $\mathbb{R}^2$ with a Lipschitz boundary and with corner points on boundary, $\partial\Omega \in \mathbb{C}^{0,1}$ and let Γ_1, Γ_2 be open disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, $\Gamma_1 \neq \emptyset$ and the 1-dimensional measure of $\partial\Omega - (\Gamma_1 \cup \Gamma_2)$ is zero. The domain Ω represents a channel filled up with a fluid, Γ_1 is a fixed wall and Γ_2 is both the input and the output of the channel.
The classical formulation of our problem is as follows:

$$-\nu\Delta\boldsymbol{u} + \nabla\mathcal{P} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{in } \Gamma_1, \tag{3}$$

$$-\mathcal{P}\mathbf{n} + \nu\frac{\partial\boldsymbol{u}}{\partial\mathbf{n}} = \mathbf{0} \quad \text{in } \Gamma_2. \tag{4}$$

Functions $\boldsymbol{u}$, $\mathcal{P}$, $\boldsymbol{f}$ are "smooth enough", $\boldsymbol{u} = (u_1, u_2)$ is velocity, $\mathcal{P}$ represents pressure, ν denotes the viscosity, $\boldsymbol{g}$ is a body force and $\mathbf{n} = (n_1, n_2)$ is an outer normal vector. The problem (1)–(4) will be called the steady Stokes problem with the mixed boundary conditions. For simplicity we suppose that $\nu = 1$ throughout this chapter.
The Dirichlet boundary condition (3) expresses a non-slip behaviour of the fluid on fixed walls of the channel. The condition (4) expresses "do nothing" boundary condition.
Existence and uniqueness of the weak solution $(\boldsymbol{u}, \mathcal{P}) \in [W^{1,2}(\Omega)]^2 \times L_2(\Omega)$ is known. Essential problems are:

- How does the smoothness of the weak solution $(\boldsymbol{u}, \mathcal{P}) \in [W^{1,2}(\Omega)]^2 \times L_2(\Omega)$ depends on the size of the angle ω_i, i.e., how regular is the weak solution?
- How depends the convergence rate of numerical methods on the regularity of the weak solution $(\boldsymbol{u}, \mathcal{P})$?

Regularity of the Stokes flows was studied by many authors for a lot of examples with different boundary conditions (see [1], [2], [4]). We give shortly the ideas and the results and refer to presented publications.
In order to get regularity results of the weak solution $(\boldsymbol{u}, \mathcal{P}) \in [W^{1,2}(\Omega)]^2 \times L_2(\Omega)$ near corner points we consider the weak solution from weighted Sobolev spaces instead of usual Sobolev spaces. We describe the standard procedure which was developed by V.A.Kondra'tev [2] and further developed by A.-M. Sändig and A.Kufner in [3] and applied in [1] to the mixed problem for the Stokes system in the following steps:

- By "localization principle" we restrict (multiplying the Stokes system (1)–(4) by cut off function) our boundary value problem to a neighborhood of a corner point O_i and consider the "modified problem" in infinite cone K_i.
- Using polar coordinates (r, ω) and the substitution $r = e^{\tau}$ and applying the complex Fourier transform with respect to τ we get the boundary value problem for the system of ordinary differential equations depending on a complex parameter λ.
- The regularity results follows from asymptotic expansion of the solution in dependence of the distribution of the eigenvalues λ.

□

Main result
Using regularity results for the weak solution $(\boldsymbol{u}, \mathcal{P})$ in corresponding weighted Sobolev spaces we prove the error estimate for the finite element approximations of velocity $\boldsymbol{u}_h$ and pressure $\mathcal{P}_h$, i.e.

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{[W^{1,2}(\Omega)]^2} + \|\mathcal{P} - \mathcal{P}_h\|_{L_2(\Omega)} = O(h^{1-\delta}) \tag{5}$$

for certain sufficiently small δ, h is parameter of discretization by means of the finite element method.

□

Conclusions
In this paper, a stationary Stokes problem equipped with mixed boundary conditions in polygonal domain have been analyzed. The regularity results, which are presented, are important for an error analysis of numerical methods, i.e., the regularity of the weak solution has a great influence over the rates of convergence for finite element methods. The main result is the proof of error estimate for finite element approximation of the weak solution in polygonal domain.

□

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