Regularity and FEM Error Estimates of Viscous Incompressible Stokes Flow in Polygonal Domains near Corners

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Abstract

We consider the stationary Stokes system with mixed boundary conditions in polygonal domain. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, $\partial\Omega \in \mathcal{C}^{0,1}$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that Γ_1 and Γ_2 are closed, sufficiently smooth, 1-dimensional measure of $\Gamma_1 \cap \Gamma_2$ is zero and 1-dimensional measure of Γ_1 is positive. We prescribe the non-slip boundary condition on Γ_1 and the boundary condition

$$-\mathcal{P}\mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0$$

on Γ_2. Here $\mathbf{u} = (u_1, u_2)$ is velocity, $\mathcal{P}$ represents pressure and $\mathbf{n} = (n_1, n_2)$ is an outer normal vector. We consider corner points on boundary, where the boundary conditions change their type. The weak solution to the Stokes system with mixed boundary conditions in a polygonal domain belongs to weighted Sobolev spaces. Regularity results are contained in [2] and [9]. The regularity results are important for an error analysis of numerical methods, i.e. the regularity of the weak solution has a great influence over the rates of convergence for finite element methods. We present finite element error estimates depending on regularity of the weak solution.

1 Introduction

Let Ω be a polygonal domain in $\mathbb{R}^2$ with a Lipschitz boundary and with corner points on boundary, $\partial\Omega \in \mathbb{C}^{0,1}$ and let Γ_1, Γ_2 be open disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, $\Gamma_1 \neq \emptyset$ and the 1-dimensional measure of $\partial\Omega - (\Gamma_1 \cup \Gamma_2)$ is zero. The domain Ω represents a channel filled up with a fluid, Γ_1 is a fixed wall and Γ_2 is both the input and the output of the channel.
The classical formulation of our problem is as follows:

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla\mathcal{P} &= \boldsymbol{f} \quad \text{in } \Omega, && (1)\\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega, && (2)\\ \boldsymbol{u} &= \mathbf{0} \quad \text{in } \Gamma_1, && (3)\\ -\mathcal{P}\mathbf{n} + \nu\frac{\partial \boldsymbol{u}}{\partial \mathbf{n}} &= \mathbf{0} \quad \text{in } \Gamma_2. && (4) \end{aligned}$$

Functions $\boldsymbol{u}$, $\mathcal{P}$, $\boldsymbol{f}$ are "smooth enough", $\boldsymbol{u} = (u_1, u_2)$ is velocity, $\mathcal{P}$ represents pressure, ν denotes the viscosity, $\boldsymbol{g}$ is a body force and $\mathbf{n} = (n_1, n_2)$ is an outer normal vector. The problem (1)–(4) will be called the steady Stokes problem with the mixed boundary conditions. For simplicity we suppose that $\nu = 1$ throughout this chapter.
The Dirichlet boundary condition (3) expresses a non-slip behaviour of the fluid on fixed walls of the channel. The condition (4) expresses "do nothing" boundary condition.

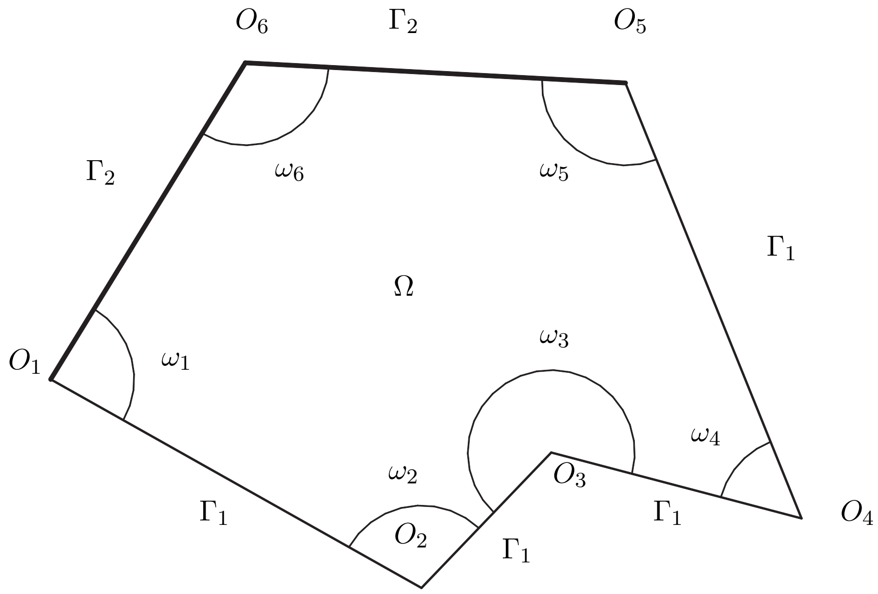

Fig. 1: Polygonal domain with corner points.

2 Weak formulation of the problem

Let

$$\mathcal{E}(\Omega) = \left\{ \boldsymbol{u} \in \mathcal{C}^\infty(\overline{\Omega})^2;\ \mathrm{div} u \equiv 0,\ \overline{\mathrm{supp}}\, u \cap \Gamma_1 \equiv \emptyset \right\}.$$

Let $V^{k,p}$ be a closure of $\mathcal{E}(\overline{\Omega})$ in the norm of $W^{k,p}(\Omega)^2$, $k \geq 0$ (k need not be an integer) and $1 \leq p < \infty$. Then $V^{k,p}$ is a Banach space with the norm of the space $W^{k,p}(\Omega)^2$. For simplicity, we denote $V^{1,2}$ and $V^{0,2}$, respectively, as V and H. Note, that V and H, respectively, are Hilbert spaces with scalar products $((.\,,.))_V$ and $((.\,,.))_H$,

$$((.\,,.))_V = ((\boldsymbol{\Phi}, \boldsymbol{\Psi}))_V = \int_\Omega \nabla\boldsymbol{\Phi} \cdot \nabla\boldsymbol{\Psi}\ d(\Omega) = \int_\Omega \frac{\partial \Phi_i}{\partial x_j}\frac{\partial \Psi_i}{\partial x_j}\ d(\Omega)$$

and

$$((.\,,.))_H = ((\boldsymbol{\Phi}, \boldsymbol{\Psi}))_H = \int_\Omega \boldsymbol{\Phi} \cdot \boldsymbol{\Psi}\ d(\Omega) = \int_\Omega \Phi_i \Psi_i\ d(\Omega)$$

and they are closed subspaces of spaces $W^{1,2}(\Omega)^2$ and $L^2(\Omega)^2$.

Definition 1 *Let $\boldsymbol{f} \in H$. Then $\boldsymbol{u}$ is called a weak solution of the Stokes problem with the mixed boundary conditions and with data $\boldsymbol{f}$ (problem (1)-(4)) if $\boldsymbol{u} \in V$ and*

$$((\boldsymbol{u}, \boldsymbol{v}))_V = ((\boldsymbol{f}, \boldsymbol{v}))_H \tag{5}$$

holds for every $\boldsymbol{v} \in V$.

Existence and uniqueness of the weak solution of (5) is known. Let us remark, that there exists some distribution $\mathcal{P} \in L_2(\Omega)$ such that

$$\nabla\mathcal{P} = \Delta\boldsymbol{u} + \boldsymbol{f}$$

in the distribution sense in Ω.

□

Essential problems are:

- How does the smoothness of the weak solution $(\boldsymbol{u}, \mathcal{P}) \in [W^{1,2}(\Omega)]^2 \times L_2(\Omega)$ depends on the size of the angle ω_i, i.e., how regular is the weak solution?
- How depends the convergence rate of numerical methods on the regularity of the weak solution $(\boldsymbol{u}, \mathcal{P})$?

3 Partition of polygonal domain

By $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, we denote the system of triangulations of $\overline{\Omega}$ with usual regularity properties from the finite element theory. $\mathcal{T}_h$ is formed by a finite number of closed triangles (or quadrilaterals) $\overline{\Omega_e}$ such that

$$\overline{\Omega} = \bigcup_{\Omega_e\in\mathcal{T}_h} \overline{\Omega_e}.$$

If Ω_{e_1}, $\Omega_{e_2} \in \mathcal{T}_h$, $\Omega_{e_1} \neq \Omega_{e_2}$, then either

$$\Omega_{e_1} \cap \Omega_{e_2} = \varnothing$$

or

$$\Omega_{e_1} \cap \Omega_{e_2} \text{ is a common vertex of } \Omega_{e_1}, \Omega_{e_2}$$

or

$$\Omega_{e_1} \cap \Omega_{e_2} \text{ is a common side of } \Omega_{e_1}, \Omega_{e_2}.$$

Denote by h_e the diameter of Ω_e and $h = \max_{\Omega_e\in\mathcal{T}_h} h_e$.
Let us assume that $\mathcal{T}_h$ is regular (see [3], [7]), i.e., there exists a constant $C > 0$ such that

$$\text{meas } \Omega_e \geq Ch^2_{\Omega_e},$$

for any $\mathcal{T}_h$ and any $\Omega_e \in \mathcal{T}_h$, where h_{Ω_e} is diam Ω_e.

4 Interpolation error

Consider now a nonempty finite dimensional subspace $V_h \subset V$. Let V_h be a finite element space of piecewise polynomial shape functions of a degree $n = k - 1$ such that

$$\lim_{h\to 0} \inf_{\boldsymbol{v}_h\in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{[W^{1,2}(\Omega)]^2} = 0.$$

Denote by I_h an interpolation operator, i.e., linear continuous operator such that $I_h : V \to V_h \subset V$, $I_h(\boldsymbol{v}_h) = \boldsymbol{v}_h$ for all $\boldsymbol{v}_h \in V_h$. In [3] and [8] have been studied approximation error estimates locally in each element. We state here the results and refer to [3], [8] for more details.
Regular partition implies the quality of the approximation $I_h\boldsymbol{u}$ of $\boldsymbol{u}$:

The following local approximation property holds for every $\boldsymbol{u} \in [W^{m,p}(\Omega)]^2$, $m \leq k$, and $\Omega_e \in \mathcal{T}_h$, provided $[W^{m,p}(\Omega)]^2 \subset [W^{l,q}(\Omega)]^2$, $0 \leq l \leq m$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$,

$$\|\boldsymbol{u} - I_h\boldsymbol{u}\|_{[W^{l,q}(\Omega_e)]^2} \leq ch^{m-l-2\left(\frac{1}{p}-\frac{1}{q}\right)} \|\nabla_m\boldsymbol{u}\|_{[L^p(\Omega_e)]^2}. \tag{6}$$

Let us now assume the general case $V_h \subset [W^{l,q}(\Omega)]^2$. The following inverse inequality holds for all $\boldsymbol{v}_h \in V_h$ (see [8])

$$\|\nabla_l\boldsymbol{v}_h\|_{[L^q(\Omega_e)]^2} \leq ch^{2\left(\frac{1}{q}-\frac{1}{q_1}\right)} \|\nabla_l\boldsymbol{v}_h\|_{[L^{q_1}(\Omega_e)]^2} \tag{7}$$

for every $\Omega_e \in \mathcal{T}_h$, $1 \leq q_1 \leq q \leq \infty$, $\Omega \subset \mathbb{R}^2$.

The relations (6) and (7) allows us to derive the global error estimates in domain Ω of FEM approximation in dependence on the regularity of the weak solution $(\boldsymbol{u}, \mathcal{P}) \in [W^{1,2}(\Omega)]^2 \times L_2(\Omega)$.

5 Regularity of the weak solution near corners

Regularity of the Stokes flows was studied by many authors for a lot of examples with different boundary conditions (see [2], [6], [9]). We give shortly the ideas and the results and refer to presented publications.

In order to get regularity results of the weak solution $(\boldsymbol{u}, \mathcal{P}) \in [W^{1,2}(\Omega)]^2 \times L_2(\Omega)$ near corner points we consider the weak solution from weighted Sobolev spaces instead of usual Sobolev spaces. The weighted Sobolev spaces are defined as follows (see [8])

$$\mathcal{V}^{k,p}(\Omega,\beta) = \left\{ u; \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u(x)|^p |x-0|^{(\beta-k+|\alpha|)p} \mathrm{dx} \right)^{\frac{1}{p}} < \infty \right\},$$

where β is arbitrary real number, $k \geq 0$.

□

It was proved by V.A.Kondra'tev (see [5]) that

$$(\boldsymbol{u}, \mathcal{P}) \in [\mathcal{V}^{2,2}(\Omega, 1+\delta)]^2 \times \mathcal{V}^{1,2}(\Omega, 1+\delta)$$

for a arbitrary small positive real number δ. The weak solution $(\boldsymbol{u}, \mathcal{P}) \in [\mathcal{V}^{2,2}(\Omega, 1+\delta)]^2 \times \mathcal{V}^{1,2}(\Omega, 1+\delta)$ can be investigated as a strong solution of (1)–(4). We describe the standard procedure which was developed by V.-A. Kondra'tev [5] and further developed by A.-M. Sändig and A. Kufner in [8] and applied in [2] to the mixed problem for the Stokes system in the following steps:

- By "localization principle" we restrict (multiplying the Stokes system (1)–(4) by cut off function) our boundary value problem to a neighborhood of a corner point O_i and consider the "modified problem" in infinite cone K_i.
- Using polar coordinates (r, ω) and the substitution $r = e^\tau$ and applying the complex Fourier transform with respect to τ we get the boundary value problem for the system of ordinary differential equations depending on a complex parameter λ.
- The regularity results follows from asymptotic expansion of the solution in dependence of the distribution of the eigenvalues λ.

Localization principle makes possible to investigate other cases of polygonal domains with many variations of different boundary conditions (see Fig. 2).

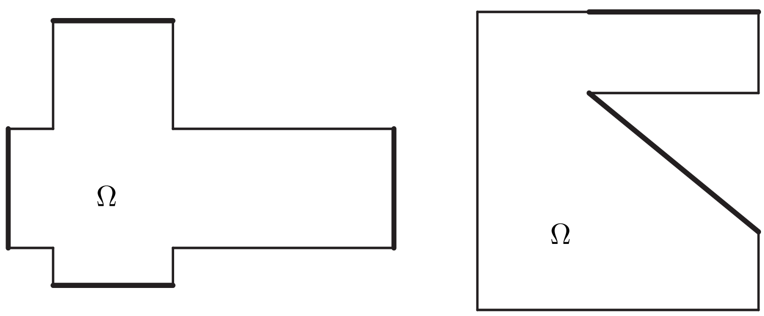

Fig. 2: Other cases of polygonal domains with different variations of boundary conditions.

□

Let us denote by ω_{DN} the maximal angle from the set of all angles corresponding to the corner points on the boundary, where boundary conditions change their type.
Analogously, let us denote by ω_{DD-NN}, the maximal angle of the remaining angles (the same type boundary conditions on the both adjacent sides of the corresponding corner point). If there is not corner point of such type or $\omega_{DD-NN} < \pi$, we set strongly $\omega_{DD-NN} = \pi$, i.e., $\omega_{DD-NN} = \pi$ is the minimal value.

For regularity of the weak solution $(\boldsymbol{u}, \mathcal{P})$ holds the following proposition (according to results obtained in [2]):

Theorem 2 *If the strip $\mu \leq Im\ \lambda < \varepsilon$, where $\mu = \delta - 1$ with arbitrary small $\varepsilon > 0$, is free of zeros of the equation (see Fig. 1)*

$$D_{DN}(\lambda) = (i\lambda)^2 \sin^2[\omega_{DN}] - \cos^2[(i\lambda)\omega_{DN}] = 0, \tag{8}$$

as well the equation (see Fig. 2)

$$D_{DD-NN}(\lambda) = (i\lambda)^2 \sin^2[\omega_{DD-NN}] - \sin^2[(i\lambda)\omega_{DD-NN}] = 0, \tag{9}$$

then

$$(\boldsymbol{u}, \mathcal{P}) \in [\mathcal{V}^{2,2}(\Omega, \delta)]^2 \times \mathcal{V}^{1,2}(\Omega, \delta) \tag{10}$$

and the following estimate holds

$$\|\boldsymbol{u}\|_{[\mathcal{V}^{2,2}(\Omega,\delta)]^2} + \|\mathcal{P}\|_{\mathcal{V}^{1,2}(\Omega,\delta)} \leq C\|\boldsymbol{f}\|_{[L_2(\Omega)]^2}. \tag{11}$$

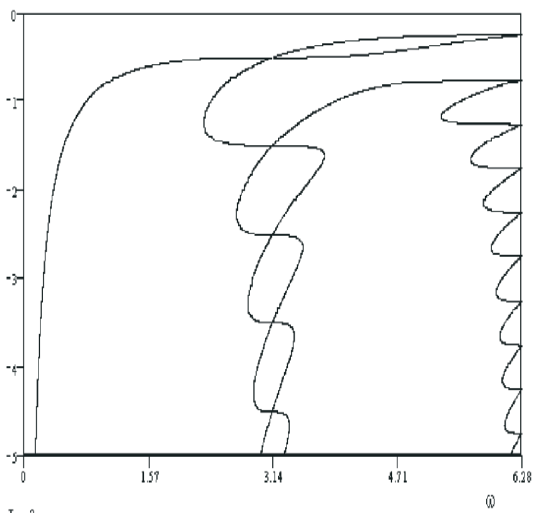

Fig. 3: $D_{DN}(\lambda) = 0$.

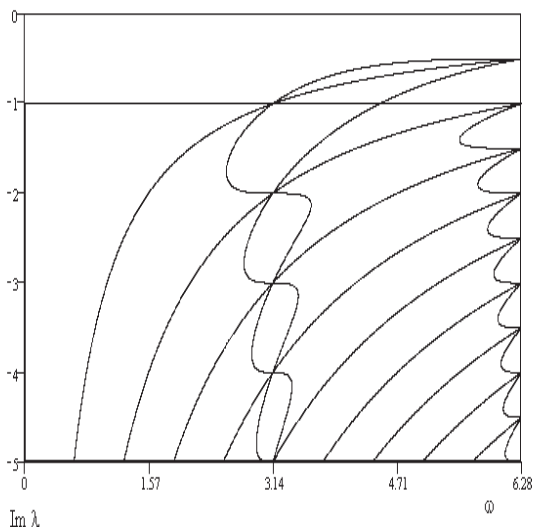

Fig. 4: $D_{DD-NN}(\lambda) = 0$.

6 FEM error analysis

6.1 Finite element aproximation

The finite dimensional subspace $V_h \subset V$ is the space of continuous functions $\boldsymbol{u}_h$ (defined on Ω)

$$\boldsymbol{u}_h = \left(\frac{\partial \psi_h}{\partial x_2}, -\frac{\partial \psi_h}{\partial x_1} \right),$$

where $\psi_h \in X_h$. For precisely definition of the space X_h see [14]. It is clear that $\operatorname{div} \boldsymbol{u}_h = 0$. This internal aproximation of V is stable and convergent provided h belongs to a regular triangulation $\mathcal{T}_h$ of Ω.

Now we are able to pronounce the aproximation of the Stokes problem (5). The problem to find $\boldsymbol{u}_h \in V_h$ such that

$$\big(\big(\boldsymbol{u}_h, \boldsymbol{v}_h\big)\big)_V = \big(\big(\boldsymbol{f}, \boldsymbol{v}_h\big)\big)_H \qquad \forall\, \boldsymbol{v}_h \in V_h, \tag{12}$$

has uniquely determined solution (see [14]). $\boldsymbol{u}_h$ is called the finite element solution (aproximation).

We have pronounced, that $\big((\,.\,,\,.)\big)_H$ is scalar product in H. From (5) and (12) follows the orthogonality relation

$$\big(\big(\,\boldsymbol{u} - \boldsymbol{u}_h,\, \boldsymbol{v}_h\big)\big)_V = 0 \qquad \forall\, \boldsymbol{v}_h \in V_h. \tag{13}$$

Further, we get for all $\boldsymbol{v}_h \in V_h$

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_V^2 &= \big(\big(\,\boldsymbol{u} - \boldsymbol{u}_h,\, \boldsymbol{u} - \boldsymbol{u}_h\big)\big)_V \\ &= \big(\big(\,\boldsymbol{u} - \boldsymbol{u}_h,\, \boldsymbol{u} - \boldsymbol{v}_h\big)\big)_V + \big(\big(\,\boldsymbol{u} - \boldsymbol{u}_h,\, \boldsymbol{v}_h - \boldsymbol{u}_h\big)\big)_V \\ &= \big(\big(\,\boldsymbol{u} - \boldsymbol{u}_h,\, \boldsymbol{u} - \boldsymbol{v}_h\big)\big)_V, \end{aligned} \tag{14}$$

since $\boldsymbol{v}_h - \boldsymbol{u}_h \in V_h$. From (14) and Schwartz inequality we get

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq \|\boldsymbol{u} - \boldsymbol{v}_h\|_V \qquad \forall\, \boldsymbol{v}_h \in V_h \tag{15}$$

and

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V = \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V. \tag{16}$$

Let I_h be a linear continuous interpolation operator $I_h : V \to V_h \subset V$ such that $I_h(\boldsymbol{v}_h) = \boldsymbol{v}_h$ (see section 4). If $\boldsymbol{u}_h \neq \boldsymbol{v}_h$ on a set with a positive measure, then

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V = \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V < \|\boldsymbol{u} - I_h \boldsymbol{u}\|_V. \tag{17}$$

□

It is well known that there exists $\mathcal{P} \in L_2(\Omega)$ which satisfy (1) in distribution sense. Let L_h be the finite dimensional subspace of $L_2(\Omega)$. We can introduce the discrete pressure $\mathcal{P}_h \in L_h$. Denote by W_h the finite dimensional subspace of $W = \big\{\Phi \in [W^{1,2}(\Omega)]^2,\ \Phi = \mathbf{0} \text{ on } \Gamma_1\big\}$. This function $\mathcal{P}_h$ is such that

$$\big(\big(\boldsymbol{u}_h, \boldsymbol{v}_h\big)\big)_V - ((\mathcal{P}_h, \operatorname{div} \boldsymbol{v}_h))_H = \big(\big(\boldsymbol{f}, \boldsymbol{v}_h\big)\big)_H \qquad \forall\, \boldsymbol{v}_h \in W_h. \tag{18}$$

Further

$$\big(\big(\boldsymbol{u}, \Phi\big)\big)_V - ((\mathcal{P}, \operatorname{div} \Phi))_H = \big(\big(\boldsymbol{f}, \Phi\big)\big)_H \qquad \forall\, \Phi \in W. \tag{19}$$

Since $W_h \subset W$ we get

$$\big(\big(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h\big)\big)_V - ((\mathcal{P} - \mathcal{P}_h, \operatorname{div} \boldsymbol{v}_h))_H = 0 \qquad \forall\, \boldsymbol{v}_h \in W_h. \tag{20}$$

Let $\boldsymbol{w}_h \in W_h$ satisfy $\operatorname{div} \boldsymbol{w}_h = \mathcal{P} - \mathcal{P}_h$. Such $\boldsymbol{w}_h$ exists (see [1]). Then

$$\|\mathcal{P} - \mathcal{P}_h\|_H^2 = \big(\big(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h\big)\big)_V \leq \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \|\mathcal{P} - \mathcal{P}_h\|_H$$

and finally

$$\|\mathcal{P} - \mathcal{P}_h\|_H \leq \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V. \tag{21}$$

6.2 Main result

Theorem 3 *Let Ω be a polygonal domain in $\mathbb{R}^2$ with the set K of singular boundary (corner) points. Let $\mathcal{T}_h$ be a family of partitions of $\overline{\Omega}$ as defined in section 3. Let the strip $\mu \leq Im\ \lambda < \varepsilon$, where $\mu = \delta - 1$ with arbitrary small $\varepsilon > 0$, is free of zeros of the equations (8) and (9) in the sense of Theorem 2. Then the finite element error can be estimated by*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{[W^{1,2}(\Omega)]^2} + \|\mathcal{P} - \mathcal{P}_h\|_{L_2(\Omega)} \leq Ch^{1-\delta}\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{22}$$

i.e.

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{[W^{1,2}(\Omega)]^2} + \|\mathcal{P} - \mathcal{P}_h\|_{L_2(\Omega)} = O(h^{1-\delta}). \tag{23}$$

Proof: From (17) it follows that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{[W^{1,2}(\Omega)]^2} \quad < \quad \|\boldsymbol{u} - I_h\boldsymbol{u}\|^2_{[W^{1,2}(\Omega)]^2} = \sum_{\Omega_e \in \mathcal{T}_h} \|\boldsymbol{u} - I_h\boldsymbol{u}\|^2_{[W^{1,2}(\Omega_e)]^2}. \tag{24}$$

We split the last term and estimate it in the following way:

$$\begin{aligned}
\|\boldsymbol{u} - I_h\boldsymbol{u}\|^2_{[W^{1,2}(\Omega_e)]^2} &\leq 2\|\boldsymbol{u}\|^2_{[W^{1,2}(\Omega_e)]^2} + 2\|I_h\boldsymbol{u}\|^2_{[W^{1,2}(\Omega_e)]^2} \\
&\leq 2\|r^{2-\delta}r^{\delta-2}\boldsymbol{u}\|^2_{[L_2(\Omega_e)]^2} + 2\|r^{1-\delta}r^{\delta-1}\nabla\boldsymbol{u}\|^2_{[L_2(\Omega_e)]^2} + 2Ch_e^{-2}\|I_h\boldsymbol{u}\|^2_{[L_2(\Omega_e)]^2} \\
&\leq 2Ch_e^{2(2-\delta)}\|r^{\delta-2}\boldsymbol{u}\|^2_{[L_2(\Omega_e)]^2} + 2Ch_e^{2(1-\delta)}\|r^{\delta-1}\nabla\boldsymbol{u}\|^2_{[L_2(\Omega_e)]^2} \\
&\qquad + 2Ch_e^{-4}\|I_h\boldsymbol{u}\|^2_{[L_2(\Omega')]^2} \\
&\leq 2Ch_e^{2(1-\delta)}\left(\|r^{\delta-2}\boldsymbol{u}\|^2_{[L_2(\Omega_e)]^2} + \|r^{\delta-1}\nabla\boldsymbol{u}\|^2_{[L_2(\Omega_e)]^2}\right) + 2Ch_e^{-4}\|I_h\boldsymbol{u}\|^2_{[\mathcal{C}(\overline{\Omega'})]^2} \\
&\leq 2Ch_e^{2(1-\delta)}\|\boldsymbol{u}\|^2_{[\mathcal{V}^{2,2}(\Omega_e,\delta)]^2} + 2Ch_e^{-4}\|\boldsymbol{u}\|^2_{[\mathcal{C}(\overline{\Omega'})]^2}
\end{aligned} \tag{25}$$

where we have used the inverse inequality (7) in the form

$$\|I_h\boldsymbol{u}\|_{[W^{1,2}(\Omega_e)]^2} \leq Ch_e^{-1}\|I_h\boldsymbol{u}\|_{[L_2(\Omega_e)]^2}$$

which is useful for our case. Remark, that the embedding

$$\mathcal{V}^{2,2}(\Omega,\delta) \hookrightarrow \mathcal{C}(\overline{\Omega})$$

holds for every $\delta \geq 0$. Then the last term in (25) we can estimate by

$$\begin{aligned}
2Ch_e^{-4}\|\boldsymbol{u}\|^2_{[\mathcal{C}(\overline{\Omega'})]^2} \leq 2Ch_e^{-4}\|\boldsymbol{u}\|^2_{[\mathcal{V}^{2,2}(\Omega',\delta)]^2} &\leq 2Ch_e^{-4}h_e^{2(3-\delta)}\|\boldsymbol{u}\|^2_{[\mathcal{V}^{2,2}(\Omega_e,\delta)]^2} \\
&\leq 2Ch_e^{2(1-\delta)}\|\boldsymbol{u}\|^2_{[\mathcal{V}^{2,2}(\Omega_e,\delta)]^2}.
\end{aligned} \tag{26}$$

From the estimates (25) and (26) we conclude

$$\|\boldsymbol{u} - I_h\boldsymbol{u}\|_{[W^{1,2}(\Omega_e)]^2} \leq Ch_e^{1-\delta}\|\boldsymbol{u}\|_{[\mathcal{V}^{2,2}(\Omega_e,\delta)]^2}.$$

Together with (11), (17) and (21) we get (22), the proof is complete.

□

7 Conclusions

In this paper, a stationary Stokes problem equipped with mixed boundary conditions in polygonal domain have been analyzed. The regularity results, which are presented, are important for an error analysis of numerical methods, i.e., the regularity of the weak solution has a great influence over the rates of convergence for finite element methods. The main result is the proof of error estimate for finite element approximation of the weak solution in polygonal domain.

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