# Conditions for existence of positive solutions of discrete delayed equations*

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## 1 Introduction and preliminaries

For given integers $s$, $q$, $s < q$ we set $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$. Possibility $s = -\infty$ or $q = \infty$ is admitted, too. We consider a scalar discrete equation

$$\Delta x(n) = -\sum_{i=0}^{k} p_i(n)x(n-i), \tag{1.1}$$

with $p_i\colon \mathbb{Z}_a^\infty \to \mathbb{R}^+ = (0,\infty)$, $i = 0, 1, \dots, k$, where $a \in \mathbb{N} := \{0, 1, \dots\}$ and $k \in \mathbb{N}$, $k > 0$. Together with discrete equation (1.1) we consider an initial problem. It is posed as follows: for a fixed $k \in \mathbb{N}$ we are seeking the solution of (1.1) satisfying $k+1$ initial conditions

$$x(n) = \varphi(n) \in \mathbb{R},\ n \in \mathbb{Z}_{a-k}^a. \tag{1.2}$$

Let us recall that the solution of the initial problem (1.1), (1.2) is defined as an infinite sequence of numbers $\{x(a) = x^0, x(a+1) = x^1, \dots, x(a+n) = x^n, x(a+n+1), x(a+n+2), \dots\}$ such that for any $n \in \mathbb{Z}_a^\infty$ the equality (1.1) holds. The existence and uniqueness of the solution of the initial problem (1.1), (1.2) is obvious for every $n \in \mathbb{Z}_a^{+\infty}$. The initial problem (1.1), (1.2) depends continuously on the initial data.

We define $\omega(n) := \{(x) : b(n) < x < c(n)\}$, where $b, c$ are real functions defined on $\mathbb{Z}_a^{+\infty}$ such that $b(n) < c(n)$ for each $n \in \mathbb{Z}_a^{+\infty}$.Let $\omega := \bigcup_{n\in\mathbb{Z}_a^{+\infty}}(n, \omega(n))$. By definition we put $\partial\omega := \{(n,x) : n \in \mathbb{Z}_a^{+\infty}, x = b(n) \text{ or } x = c(n)\}$. Our aim is to establish a set of sufficient conditions with respect to the righ-hand side of equation (1.1) in order to guarantee the existence of at least one solution $x = x(n)$ defined on $\mathbb{Z}_a^{+\infty}$ such that $(n, x(n)) \subset (n, \omega(n))$ for each $n \in \mathbb{Z}_a^{+\infty}$.

Let $B_1 := \{(n,x) : n \in \mathbb{Z}_a^{+\infty},\ x = b(n)\}$, $B_2 := \{(n,x) : n \in \mathbb{Z}_a^{+\infty},\ x = c(n)\}$. Obviously $\partial\omega = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$.

**Lemma 1.1.** *A point $(n,x) \in B_1 \cup B_2$ with $n \in \mathbb{Z}_a^{+\infty}$ is the point of the type of strict egress for the set $\omega$ with respect to the discrete equation* (1.1) *if and only if*

$$-\sum_{i=1}^{k} p_i(n)x(n-i) - p_0(n)b(n) - b(n+1) + b(n) < 0 \tag{1.3}$$

*Preliminary version

*for every* $x(n-i) \in \omega(n-i)$*,* $i = 1, 2, \ldots, k$ *in the case when* $(n, x) \in B_1$*, and*

$$-\sum_{i=1}^{k} p_i(n)x(n-i) - p_0(n)c(n) - c(n+1) + c(n) > 0 \tag{1.4}$$

*for every* $x(n-i) \in \omega(n-i)$*,* $i = 1, 2, \ldots, k$ *in the case when* $(n, x) \in B_2$*.*

**Theorem 1.2.** *Let inequalities* (1.3)*,* (1.4) *be valid for every* $n \in \mathbb{Z}_a^{+\infty}$ *and every* $x(n-i) \in \omega(n-i)$*,* $i = 1, \ldots, k$*. Then there exists an initial problem*

$$x^*(a-m) = x_m^* \in \omega(a-m), \; m = 0, 1, \ldots, k \tag{1.5}$$

*such that the corresponding solution* $x = x^*(n)$ *of equation* (1.1) *satisfies for every* $n \in \mathbb{Z}_a^{+\infty}$ *the inequalities*

$$b(n) < x^*(n) < c(n). \tag{1.6}$$

## 2 Conditions for existence of a positive solution of (1.1)

**Theorem 2.1.** *Existence of a function* $\nu\colon \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}^+$ *satisfying*

$$\Delta\nu(n) \le -\sum_{i=0}^{k} p_i(n)\nu(n-i) \tag{2.1}$$

*for every* $n \in \mathbb{Z}_a^{+\infty}$ *is necessary and sufficient for existence of a solution* $x\colon \mathbb{Z}_{a-k}^{+\infty} \to \mathbb{R}^+$ *of* (1.1)*. Moreover,* $x(n) < \nu(n)$ *holds on* $\mathbb{Z}_{a-k}^{+\infty}$*.*

We consider a linear discrete equation

$$\Delta x(n) = -\sum_{i=0}^{k} P_i(n)x(n-i) \tag{2.2}$$

with $P_i\colon \mathbb{Z}_a^{\infty} \to \mathbb{R}_+ = [0, \infty)$, $i = 0, 1, \ldots, k$ and $\sum_{i=0}^{k} P_i(n) > 0$.

**Theorem 2.2.** *Let* $x = \mu\colon \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}^+$ *be a solution of* (1.1) *and* $P_i(n) \le p_i(n)$*,* $i = 1, 2, \ldots, k$*,* $n \in \mathbb{Z}_{a-k}^{\infty}$*. Then the equation* (2.2) *has a positive solution* $x = x(n)$ *on* $\mathbb{Z}_{a-k}^{\infty}$ *and, moreover,* $x(n) < \mu(n)$*,* $n \in \mathbb{Z}_{a-k}^{\infty}$ *holds.*

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