Conditions for existence of positive solutions of discrete delayed equations*

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Abstract

In the contribution we discuss existence of positive solutions of a discrete linear delayed equation. Some existence are established. Upper bound for such solutions is given as well.

Keywords and Phrases: Discrete delayed equation, positive solution, upper bound
AMS Subject Classification: 39A10, 39A11

1 Introduction and preliminaries

For given integers s, q, $s < q$ we set $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$. Possibility $s = -\infty$ or $q = \infty$ is admitted, too. Throughout this paper, using notation $\mathbb{Z}_s^q$ or another one with couple integers s, q we suppose $s \le q$. We consider a scalar discrete equation

$$\Delta x(n) = -\sum_{i=0}^{k} p_i(n)x(n-i), \tag{1.1}$$

with $p_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}^+ = (0, \infty)$, $i = 0, 1, \dots, k$, where $a \in \mathbb{N} := \{0, 1, \dots\}$ and $k \in \mathbb{N}$, $k > 0$. Together with discrete equation (1.1) we consider an initial problem. It is posed as follows: for a fixed $k \in \mathbb{N}$ we are seeking the solution of (1.1) satisfying $k+1$ initial conditions

$$x(n) = \varphi(n) \in \mathbb{R},\ n \in \mathbb{Z}_{a-k}^a. \tag{1.2}$$

Let us recall that the solution of the initial problem (1.1), (1.2) is defined as an infinite sequence of numbers

$$\{x(a) = x^0, x(a+1) = x^1, \dots, x(a+n) = x^n, x(a+n+1), x(a+n+2), \dots\}$$

such that for any $n \in \mathbb{Z}_a^\infty$ the equality (1.1) holds. The existence and uniqueness of the solution of the initial problem (1.1), (1.2) is obvious for every $n \in \mathbb{Z}_a^{+\infty}$. The initial problem (1.1), (1.2) depends continuously on the initial data.
We define

$$\omega(n) := \{(x) : b(n) < x < c(n)\},$$

*Preliminary version

where b, c are real functions defined on $\mathbb{Z}_a^{+\infty}$ such that $b(n) < c(n)$ for each $n \in \mathbb{Z}_a^{+\infty}$.
Let

$$\omega := \bigcup_{n \in \mathbb{Z}_a^{+\infty}} (n, \omega(n)).$$

By definition we put

$$\partial\omega := \{(n, x) : n \in \mathbb{Z}_a^{+\infty}, x = b(n) \text{ or } x = c(n)\}.$$

Our aim is to establish a set of sufficient conditions with respect to the righ-hand side of equation (1.1) in order to guarantee the existence of at least one solution $x = x(n)$ defined on $\mathbb{Z}_a^{+\infty}$ such that $(n, x(n)) \subset (n, \omega(n))$ for each $n \in \mathbb{Z}_a^{+\infty}$.
Let

$$\begin{array}{l} B_1 := \{(n, x) : n \in \mathbb{Z}_a^{+\infty},\ x = b(n)\}, \\ B_2 := \{(n, x) : n \in \mathbb{Z}_a^{+\infty},\ x = c(n)\}. \end{array},$$

Obviously $\partial\omega = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$. Finally, we define functions $U_1, U_2 \colon \mathbb{Z}_a^{+\infty} \times \mathbb{R} \to \mathbb{R}$ as

$$U_1(n, x) := x - b(n),$$

$$U_2(n, x) := x - c(n).$$

2 A nonlinear principle

Let us consider the scalar discrete equation

$$\Delta u(n) = f(n, u(n), u(n-1), \dots, u(n-k)), \tag{2.1}$$

where $f \colon \mathbb{Z}_a^{+\infty} \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $k \in \mathbb{N}$. Together with discrete equation (2.1) we consider an initial problem. It is posed as follows: for a given $s \in \mathbb{N}$ we are seeking the solution of (2.1) satisfying $k+1$ initial conditions

$$u(a + s - m) = u^{s-m} \in \mathbb{R},\ m = 0, 1, \dots, k \tag{2.2}$$

with prescribed constants u^{s-m}. Let us recall that the solution of the initial problem (2.1), (2.2) is defined as an infinite sequence of numbers

$$\begin{gathered} \{u(a+s) = u^s, u(a+s+1) = u^{s+1}, \dots, u(a+s+k) = u^{s+k}, \\ u(a+s+k+1), u(a+s+k+2), \dots\} \end{gathered}$$

such that for any $n \in \mathbb{Z}_{a+s}^{+\infty}$ the equality (2.1) holds. The existence and uniqueness of the solution of the initial problem (2.1), (2.2) is obvious for every $k \in \mathbb{Z}_{a+s}^{+\infty}$. If f is continuous with respect to last $(k+1)$ coordinates, then the initial problem (2.1), (2.2) depends continuously on the initial data.

Definition 2.1. *The full difference*

$$\Delta U_1(n, u)|_{(n-m, u) \in (n-m, \omega(n-m)), m=1,\dots,k,\ (n,u) \in B_1}$$

of the function $U_1(n, u)$ for a fixed $n \in \mathbb{Z}_a^{+\infty}$ with respect to the discrete equation (2.1) *and the sets B_1 and ω is defined as*

$$\begin{aligned} &\Delta U_1(n, u)|_{(n-m, u) \in (n-m, \omega(n-m)), m=1,\dots,k,\ (n,u) \in B_1} := \\ &f(n, b(n), u_1, \dots, u_k) - b(n+1) + b(n), \end{aligned}$$

where $u_1 \in \omega(n-1), \dots, u_k \in \omega(n-k)$ is assumed.

Definition 2.2. *The full difference*

$$\Delta U_2(n,u)|_{(n-m,u)\in(n-m,\omega(n-m)),m=1,\dots,k,\,(n,u)\in B_2}$$

of the function $U_2(n,u)$ *for a fixed* $n \in \mathbb{Z}_a^{+\infty}$ *with respect to the discrete equation* (2.1) *and the sets* B_2 *and* ω *is defined as*

$$\Delta U_2(n,u)|_{(n-m,u)\in(n-m,\omega(n-m)),m=1,\dots,k,\,(n,u)\in B_2} := \\ f(n,c(n),u_1,\dots,u_k) - c(n+1) + c(n),$$

where $u_1 \in \omega(n+1),\dots,u_k \in \omega(n-k)$ *is assumed.*

In the following text we will abbreviate corresponding notation and put

$$\Delta U_1^*(n,u) \equiv \Delta U_1(n,u)|_{(n-m,u)\in(n-m,\omega(n-m)),m=1,\dots,k,\,(n,u)\in B_1},$$

$$\Delta U_2^*(n,u) \equiv \Delta U_2(n,u)|_{(n-m,u)\in(n-m,\omega(n-m)),m=1,\dots,k,\,(n,u)\in B_2}.$$

Definition 2.3. *A point* $(n,u) \in B_1$ *with* $n \in \mathbb{Z}_a^{+\infty}$ *is called* the point of the type of strict egress *for the set* ω *with respect to the discrete equation* (2.1) *if*

$$\Delta U_1^*(n,u) < 0$$

for every $u_1 \in \omega(n-1),\dots,u_k \in \omega(n-k)$.

Definition 2.4. *A point* $(n,u) \in B_2$ *with* $n \in \mathbb{Z}_a^{+\infty}$ *is called* the point of the type of strict egress *for the set* ω *with respect to the discrete equation* (1.1) *if*

$$\Delta U_2^*(n,u) > 0$$

for every $u_1 \in \omega(n-1),\dots,u_k \in \omega(n-k)$.

The following lemma is obvious.

Lemma 2.5. *A point* $(n,u) \in B_1 \cup B_2$ *with* $n \in \mathbb{Z}_a^{+\infty}$ *is the point of the type of strict egress for the set* ω *with respect to the discrete equation* (2.1) *if and only if*

$$f(n,b(n),u_1,\dots,u_k) - b(n+1) + b(n) < 0 \tag{2.3}$$

for every $u_1 \in \omega(n-1),\dots,u_k \in \omega(n-k)$ *in the case when* $(n,u) \in B_1$, *and*

$$f(n,c(n),u_1,\dots,u_k) - c(n+1) + c(n) > 0 \tag{2.4}$$

for every $u_1 \in \omega(n-1),\dots,u_k \in \omega(n-k)$ *in the case when* $(n,u) \in B_2$.

Now we are ready to formulate a nonlinear result concerning the existence of a solution of (2.1) with the graph lying in the set ω (see [2, 3]).

Theorem 2.6. *Let* $f\colon \mathbb{Z}_a^{+\infty} \times \mathbb{R}^{n+1} \to \mathbb{R}$ *be continuous with respect to last* $(k+1)$ *coordinates. If, moreover, the inequalities* (2.3), (2.4) *hold for every* $n \in \mathbb{Z}_a^{+\infty}$ *and every*

$$u_1 \in \omega(n-1),\dots,u_k \in \omega(n-k),$$

then there exists an initial problem

$$u^*(a-m) = u_m^* \in \omega(a-m),\ m = 0,1,\dots,k \tag{2.5}$$

such that the corresponding solution $u = u^*(n)$ *of equation* (2.1) *satisfies for every* $n \in \mathbb{Z}_a^{+\infty}$ *the inequalities*

$$b(n) < u^*(n) < c(n). \tag{2.6}$$

3 The linear case

We adapt definitions and results of the previous Section 2 to our equation (1.1).

Definition 3.1. *The full difference*

$$\Delta U_1(n,x)|_{(n,x)\in B_1}$$

of the function $U_1(n,x)$ *for a fixed* $n\in\mathbb{Z}_a^{+\infty}$ *with respect to the discrete equation* (1.1) *and the sets* B_1 *and* ω *is defined as*

$$\Delta U_1(n,x)|_{(n,x)\in B_1} = -\sum_{i=1}^{k} p_i(n)x(n-i) - p_0(n)b(n) - b(n+1) + b(n)$$

where $x(n-i)\in\omega(n-i)$, $i=1,2,\dots,k$ *is assumed.*

Definition 3.2. *The full difference*

$$\Delta U_2(n,x)|_{(n,x)\in B_2}$$

of the function $U_2(n,x)$ *for a fixed* $n\in\mathbb{Z}_a^{+\infty}$ *with respect to the discrete equation* (1.1) *and the sets* B_2 *and* ω *is defined as*

$$\Delta U_2(n,x)|_{(n,x)\in B_2} = -\sum_{i=1}^{k} p_i(n)x(n-i) - p_0(n)c(n) - c(n+1) + c(n)$$

where $x(n-i)\in\omega(n-i)$, $i=1,2,\dots,k$ *is assumed.*

Definition 3.3. *A point* $(n,x)\in B_1$ *with* $n\in\mathbb{Z}_a^{+\infty}$ *is called* the point of the type of strict egress *for the set* ω *with respect to the discrete equation* (1.1) *if*

$$\Delta U_1(n,x) < 0$$

for every $x(n-i)\in\omega(n-i)$, $i=0,1,2,\dots,k$.

Definition 3.4. *A point* $(n,x)\in B_2$ *with* $n\in\mathbb{Z}_a^{+\infty}$ *is called* the point of the type of strict egress *for the set* ω *with respect to the discrete equation* (1.1) *if*

$$\Delta U_2(n,x) > 0$$

for every $x(n-i)\in\omega(n-i)$, $i=0,1,2,\dots,k$.

Lemma 3.5. *A point* $(n,x)\in B_1\cup B_2$ *with* $n\in\mathbb{Z}_a^{+\infty}$ *is the point of the type of strict egress for the set* ω *with respect to the discrete equation* (1.1) *if and only if*

$$-\sum_{i=1}^{k} p_i(n)x(n-i) - p_0(n)b(n) - b(n+1) + b(n) < 0 \tag{3.1}$$

for every $x(n-i)\in\omega(n-i)$, $i=1,2,\dots,k$ *in the case when* $(n,x)\in B_1$, *and*

$$-\sum_{i=1}^{k} p_i(n)x(n-i) - p_0(n)c(n) - c(n+1) + c(n) > 0 \tag{3.2}$$

for every $x(n-i)\in\omega(n-i)$, $i=1,2,\dots,k$ *in the case when* $(n,x)\in B_2$.

Theorem 3.6. *Let inequalities* (3.1), (3.2) *be valid for every* $n \in \mathbb{Z}_a^{+\infty}$ *and every*

$$x(n-i) \in \omega(n-i), \;\; i = 1, \dots, k.$$

Then there exists an initial problem

$$x^*(a-m) = x_m^* \in \omega(a-m), \; m = 0, 1, \dots, k \tag{3.3}$$

such that the corresponding solution $x = x^*(n)$ *of equation* (1.1) *satisfies for every* $n \in \mathbb{Z}_a^{+\infty}$ *the inequalities*

$$b(n) < x^*(n) < c(n). \tag{3.4}$$

4 Conditions for existence of a positive solution of (1.1)

Theorem 4.1. *Existence of a function* $\nu\colon \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}^+$ *satisfying*

$$\Delta\nu(n) \le -\sum_{i=0}^{k} p_i(n)\nu(n-i) \tag{4.1}$$

for every $n \in \mathbb{Z}_a^{+\infty}$ *is necessary and sufficient for existence of a solution* $x\colon \mathbb{Z}_{a-k}^{+\infty} \to \mathbb{R}^+$ *of* (1.1). *Moreover,* $x(n) < \nu(n)$ *holds on* $\mathbb{Z}_{a-k}^{+\infty}$.

Proof. (*Necessity.*) Let $x\colon \mathbb{Z}_{a-k}^{+\infty} \to \mathbb{R}^+$ be a solution of (1.1). We define $\nu(n) := x(n)$, $n \in \mathbb{Z}_{a-k}^{+\infty}$. Then ν is positive on $\mathbb{Z}_{a-k}^{\infty}$ and obviously satisfies the inequality (4.1).

(*Sufficiency.*) Let ν is a positive on $\mathbb{Z}_{a-k}^{\infty}$ function satisfying the inequality (4.1). Let $\chi(n)$, $n \in \mathbb{Z}_a^{\infty}$ be the set of functions $\lambda\colon \mathbb{Z}_{n-k}^{n} \to \mathbb{R}$ such that

$$0 < \lambda(i) < \nu(i), \; i \in \mathbb{Z}_{n-k}^{n-1}$$

and either $\lambda(n) = 0$ or $\lambda(n) = \nu(n)$. So, $\lambda(i)$ have the same sence as the functions $\omega(i)$ above. Further, define the function

$$W(n,x) \equiv x(n) \cdot (x(n) - \nu(n)), \quad n \in \mathbb{Z}_{a-k}^{a} \tag{4.2}$$

and we find the sign of the full difference of this function with respect to (1.1) and $\chi(n)$ for each $n \in \mathbb{Z}_a^{\infty}$. Obviously

$$\Delta W(n,x) = \Delta x(n) \cdot (x(n) - \nu(n)) + x(n) \cdot \Delta(x(n) - \nu(n))$$

and

$$\Delta W(n,x) = \left(-\sum_{i=0}^{k} p_i(n)x(n-i)\right)(x(n) - \nu(n)) + x(n)\left(-\sum_{i=0}^{k} p_i(n)x(n-i) - \Delta\nu(n)\right).$$

For each $\lambda \in \chi$, such that $\lambda(n) = 0$, we obtain

$$\Delta W(n,x)|_{x=\lambda} = \left(-\sum_{i=1}^{k} p_i(n)x(n-i)\right)(-\nu(n)) = \nu(n) \cdot \left(\sum_{i=1}^{k} p_i(n)x(n-i)\right) > 0.$$

Because $\lambda(n) = b(n)$ and $b(n) = b(n+1) = 0$, we have

$$-\sum_{i=1}^{k} p_i(n)x(n-i) < 0,$$

$$-\sum_{i=1}^{k} p_i(n)x(n-i) - p_0 \cdot 0 - 0 + 0 < 0,$$

$$-\sum_{i=1}^{k} p_i(n)x(n-i) - p_0 b_0 - b(n+1) - b(n) < 0,$$

so inequalities (3.1) holds.

Moreover, for each $\lambda \in \chi$, such that $\lambda(n) = \nu(n)$, we get for $\nu(n) = c(n), \nu(n+1) = c(n+1)$

$$\Delta W(n,x)|_{x=\lambda} = \nu(n)\left(-\sum_{i=0}^{k} p_i(n)x(n-i) - \Delta\nu(n)\right) =$$

$$\nu(n)\left(-\sum_{i=0}^{k} p_i(n)x(n-i) - p_0(n)c(n) - c(n+1) + c(n)\right) > 0.$$

So inequalities (3.2) holds.

Therefore, in both cases, for $n \in \mathbb{Z}_a^{+\infty}$, the following is true:

$$\Delta W(n,x)|_{x=\lambda} > 0. \tag{4.3}$$

Now, in accordance with Theorem 3.6 there exists a solution of (1.1)

$$x = \tilde{x}(n), \ \ n \in \mathbb{Z}_a^{+\infty},$$

such that

$$\tilde{x}(n) \in \chi(n).$$

From the form of the set $\chi(n)$ it follows that this solution is positive and less then $\nu(n)$ on $\mathbb{Z}_{a-k}^{a}$. □

We consider a linear discrete equation

$$\Delta x(n) = -\sum_{i=0}^{k} P_i(n)x(n-i) \tag{4.4}$$

with $P_i \colon \mathbb{Z}_a^{\infty} \to \mathbb{R}_+ = [0,\infty)$, $i = 0,1,\dots,k$ and $\sum_{i=0}^{k} P_i(n) > 0$.

Theorem 4.2. *Let $x = \mu \colon \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}^+$ be a solution of (1.1) and $P_i(n) \le p_i(n)$, $i = 1,2,\dots,k$, $n \in \mathbb{Z}_{a-k}^{\infty}$. Then the equation (4.4) has a positive solution $x = x(n)$ on $\mathbb{Z}_{a-k}^{\infty}$ and, moreover, $x(n) < \mu(n)$, $n \in \mathbb{Z}_{a-k}^{\infty}$ holds.*

Proof. It follows from Theorem 4.1. The function μ has the sense as the function ν ih the proof of Theorem 4.1.

□

Acknowledgement. The first author was supported by the Grant 201/04/0580 of Czech Grant Agency (Prague) and the second author was supported by the Council of Czech Government MSM 00216 30503.

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