Hypergroupoids on Partially Ordered Sets

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Abstract: This paper links up the papers [1], [2], [3], [4], [5], [7] and [11]. The hyperoperation on partially-ordered sets is introduced. It is proved that partial ordered sets with such hyperoperation form semihypergroups. Further, the relation of equivalence and also congruence is studied at the end of the first part. The second part is applied to the studies of a special congruence and the third part deals with distinguishing subsets on the partial ordered semihypergroups. The fourth part contains some examples, which describe the studied operations and relations on a concrete partial ordered set.

Key words. Semihypergroups, binary hyperoperation, equivalence and congruence on semihypergroup. Distinguishing and weakly-distinguishing subsets of semihypergroups.

1. The Leading Article

1.1 Definition *A hypergroupoid* (or *a multigroupoid*) is a pair $(M, \circ)$ where M is a nonempty set and $\circ : M \times M \to \mathcal{P}^*(M)$ is a binary hyperoperation called also a multioperation. ($\mathcal{P}^*(M)$ is the system of all nonempty subsets of M).
A semihypergroup is an associative hypergroupoid, i.e. hypergroupoid satisfying the equality $(a \circ b) \circ c = a \circ (b \circ c)$ for every triple a, b, $c \in M$.

1.2 Introduction we denote by $\mathcal{M}$ a partially ordered set M with the ordering $\leq$ with the greatest element I which will be inscribed in the next part of this article with $\mathcal{M} = (M, \leq, I)$

1.3 Definition By the length of a chain consisting of $r + 1$ elements that is of the form

$$x_0 \prec x_1 \prec x_2 \prec \ \ldots \prec x_r \qquad [x_0, x_r]$$

(where the notation $x_i \prec x_{i+1}$ means that the element x_i is covered by the element x_{i+1} - see [12]) we mean the non-negative number r. We define the length of a partial ordered set $(\mathcal{M} = (M, \leq, I)$ as

$$max\{r_j \mid r_i, j \in J \text{ as the lengths of chains in } M\}.$$

(This definition is opposite to the definition of the length of ordered set in [12]). We shall devote attention to partially ordered sets of finite length.

1.4 Definition We introduce for every element $u \in M$ a subset $U \subseteq M$ as follows: $U = \{u_i \mid u_i \geq u\}$ and on $\mathcal{M} = (M, \leq, I)$ we define for arbitrary $x, y \in M$ the binary hyperoperation $\circ$ as follows:

$$x \circ y = \{min(X \cap Y)\}.$$

We denote then the set $\mathcal{M}$ with such defined binary operation with $\mathcal{M} = (\,M \leq, \circ, I)$.

1.5 Definition - Remark We introduce the following very important concept. A subset Di of $\mathcal{M} = (\,M \leq, \circ, I)$ is called *dual ideal* of $\mathcal{M}$ if Di satisfies the following condition:

For $x, y \in Di$ the relation $x \circ y \subset Di$ holds.

The subset U defined in 1.4 is the dual ideal of the element $u \in \mathcal{M} = (\,M \leq, \circ, I)$.

1.6 Lemma The hyperoperation of multiplication $\circ$ on $(\mathcal{M} = (\,M \leq, \circ, I)$ is idempotent.

Proof. It is obvious that $X \cap X = X$ and hence $min(X \cap X) = minX$. Hence we receive $x \circ x = \{x\}$.

1.7 Lemma The binary hyperoperation $\circ$ on $\mathcal{M} = (\,M \leq, \circ, I)$ is commutative.

Proof. The proposition follows directly from the definition of binary hyperoperation $\circ$. We have $X \cap Y = Y \cap X$ and hence $\{min(X \cap Y)\} = \{min(Y \cap X)\}$.

1.8 Theorem $\mathcal{M} = (\,M, \leq, \circ, I)$ is a commutative hypergroupoid.

Proof. The theorem follows directly from 1.6 and 1.7.

1.9 Theorem Every upper-ideal of $\mathcal{M} = (\,M \leq, I)$ is identical with the dual-ideal of the commutative hypergroupoid. $\mathcal{M} = (\,M \leq, \circ, I)$.

Proof. The proof follows from the definition of the operation $\circ$ for the products $a \circ a$ and $x \circ y$, where a, x, $y \in M$, $a < x, a < y$.

1.10 Remark The binary hyperoperation $\circ$ defined on partially ordered sets is not associative. It is obvious from the following example.

1.11 Example

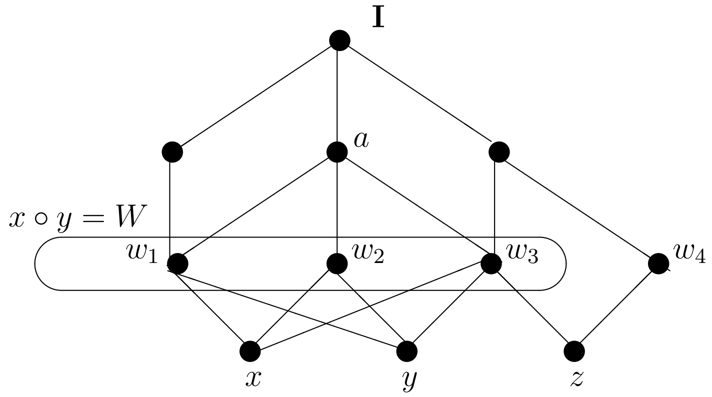

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Figure 1

We show in this example that the hyperoperation $\circ$ given on ordered set on Figure 1 is not associative. We prove that $(x \circ y) \circ z \neq x \circ (y \circ z)$. We denote $x \circ y$ as W. ¿From the definition of the hyperoperation $\circ$ the set $W = \{w_1, w_2, w_3\}$.

$W \circ z = \{w_1 \circ z\} \cup \{w_2 \circ z\} \cup \{w_1 \circ z\} = \{a\} \cup \{a\} \cup \{w_3\} = \{a. w_3\}$.

Similarly we denote $y \circ z$ as U. Then $U = \{w_3\}$ and $x \circ U = x \circ w_3 = \{w_3\}$. Hence we proved Remark 1.10.

2. Congruences on ordered semihypergroups with the least element

2.1 Introduction In this chapter we suppose the ordered hypergroupoids with the least and the greatest elements, $\mathcal{M} = (\, M \leq, \circ, 0, I)$.

2.2 Definition A congruence on an ordered hypergroupoid $\mathcal{M}$ is called a relation of equivalence ρ on $\mathcal{M}$ such that for every quaternion of elements $a_1, a_2, b_1, b_2 \in M$ for which $a_1 \rho\, b_1$, $a_2 \rho\, b_2$ the following holds: For every $x \in a_1 \circ a_2$ there exists $y \in b_1 \circ b_2$ and for every $y' \in b_1 \circ b_2$ there exists $x' \in a_1 \circ a_2$ with the property $x \rho\, y$ and $x' \rho\, y'$. See [4] p.151 and [10].

2.3 Definition - Remark Let ρ be a congruence relation on hypergroupoid $\mathcal{M}$. The symbol M/ρ denotes a set of classes of the congruence ρ. We define binary operation $\diamond$ onto this set as follows. We assign to the double of classes $\rho(a), \rho(b) \in M/\rho$ the product $\diamond$ as the class $\rho(a) \diamond \rho(b) = \rho(a \circ b)$. The class $\rho(a \circ b)$ does not depend onto the choice of the elements $a, b \in M$ but only onto the classes $\rho(a)$, $\rho(b)$. Hence we have for $a^{'} \in \rho(a)$, $b^{'} \in \rho(b)$ the equation $\rho(a \circ b) = \rho(a^{'} \circ b^{'})$.

2.4 Definition Let $\mathcal{M} = (\, M, \leq, \circ, I)$ be an ordered hypergroupoid, L a subset of M. We say that the elements x, y from M are in the relation $\Xi_{(M,L)}$ on ordered hypergroupoid $\mathcal{M}$ iff $x \circ u \subseteq L$ is equivalent to $y \circ u \subseteq L$ for all elements $u \in M$.

2.5 Theorem Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid. Let $L \subseteq M$. Then $\Xi_{(M,L)}$ be a congruence relation on $\mathcal{M}$. If $x \in L$ and the pair $(x, y) \in \Xi_{(M,L)}$ then $y \in L$.

Proof. From the definition of the relation $\Xi_{(M,L)}$ follows that it is an equivalence relation. It is obvious that the relation $\Xi_{(M,L)}$ is reflexive and commutative. We prove the transitivity. Let be $x, y, z, u \in M$. The relation $(x \circ u)\ \Xi_{(M,L)}\ (y \circ u)$ implies $x \circ u \subseteq L$ is equivalent to $y \circ u \subseteq L$. Similarly $(y \circ u)\ \Xi_{(M,L)}\ (z \circ u)$ implies $y \circ u \subseteq L$ is equivalent to $z \circ u \subseteq L$. Hence $(x \circ u)\ \Xi_{(M,L)}\ (z \circ u)$.

If $x \in L$ and $(x, y) \in \Xi_{(M,L)}$ then we have $x \circ 0 = x$ and at the same time $y \circ 0 = y$

2.6 Remark We show that the congruence $\Xi_{(M,L)}$ satisfies the definition 2.2.

Proof. We put $a_1 = x$, $b_1 = y$, $a_2 = v = b_2$. Let $x\ \Xi_{(M,L)}\ y$, evidently $u\ \Xi_{(M,L)}\ u$ and hence $(x \circ u)\ \Xi_{(M,L)}\ (y \circ u)$ that means $x \circ u \subseteq L$ is equivalent to $y \circ u \subseteq L$. The sets $x \circ u$, $y \circ u$ are not void (M has the maximal element) and for every element $x' \in x \circ u$ there exists $y' \in y \circ u$ such that $x'\ \Xi_{(M,L)}\ y'$. The operation $\circ$ is commutative therefore $\Xi_{(M,L)}$ satisfies the definition 2.2.

2.6 Corollary Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, $L \subseteq M$ its subset, f canonical surjection of M onto $M/\Xi_{(M,L)}$. Then $f^{-1}\{f(L)\} = L$.

Proof. Evidently $L \subseteq f^{-1}\{f(L)\} = L$. Let us consider $x \in f^{-1}\{f(L)\}$. Hence $f(x) \in f(L)$ which implies the existence of the element $y \in L$ with the property $f(x)\ f(y)$, which is equivalent to $(x, y) \in \Xi_{(M,L)}$. With respect to 2.5 we have $x \in L$ and hence $f^{-1}\{f(L)\} \subseteq L$.

2.7 Corollary Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, $L \subseteq M$ its subset. Then $L = \bigcup X$ where $X \in M/\Xi_{(M,L)}$, $X \cap L \neq \emptyset$.

2.8 Lemma Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid. Let Θ be a congruence relation onto $\mathcal{M}$ and

$$L = \bigcup_{i \in I} H_i,$$

where $H_i \in M/\Theta$ for $i \in I$. Then $\Theta \subseteq \Xi_{(M,L)}$.

Proof. Let $x\Theta y$ For all $u \in M$ $x \circ u\ \Theta\ y \circ u$. From the relation $L = \bigcup_{i \in I} H_i$ simultaneously $x \circ u$, $y \circ u \subseteq L$ or $x \circ u$, $y \circ u \nsubseteq L$. Hence $(x, y) \in \Xi_{(M,L)}$.

2.9 Lemma Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid. Let Θ be a congruence relation onto $\mathcal{M}$ and

$$L = \bigcup_{i \in I} H_i \subseteq M,$$

where $H_i \in M/\Theta$ for $i \in I$. Then

$$\bigcap_{i \in I} \Xi_{(M,H_i)} \subseteq \Xi_{(M,L)}$$

.

Proof. Let

$$(x, y) \in \bigcap_{i \in I} \Xi_{(M,H_i)},\ x, y \in M$$

. Then $(x, y) \in \Xi_{(M,H_i)}$ for all $i \in I$. It means that the products $x \circ u,\ y \circ u$ lapse either into H_i or into $M - H_i$ for all $u \in M$ and all $i \in I$. If $x \circ u,\ y \circ u \subseteq M - H_i$, then either simultaneously

$$x \circ u,\ y \circ u \subseteq M - \bigcup_{i \in I} H_i$$

or simultaneously $x \circ u,\ y \circ u \subseteq H_j$ where $j \in I, j \neq i$. In the opposite case $(x, y) \in \Xi_{(M,H_j)}$ holds and it is a contradiction with the relation $(x, y) \in \Xi_{(M,H_i)}$ for all $i \in I$.
Thus $x \circ u,\ y \circ u$ lapse into

$$\bigcup_{i \in I} H_i = L$$

or into the complement $M - L$. Hence $(x, y) \in \Xi_{(M,L)}$.

2.10 Lemma Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, Θ congruence relation on $\mathcal{M}$ and $H_1, H_2 \in M/\theta$ and let $H_2 \in M/\Xi_{(M,H_1)}$. Then $\Xi_{(M,H_1)} \subseteq \Xi_{(M,H_2)}$.
Proof. Let $x, y, u \in M$ be such that $(x, y) \in \Xi_{(M,H_1)}$ and $x \circ u \subseteq H_2$. By reason that $H_2 \in M/\Xi_{(M,H_1)}$ the relation $y \circ u \subseteq H_2$ holds true. Analogously from $y \circ u \subseteq H_2$ follows $x \circ u \subseteq H_2$. Hence $x \circ u \subseteq H_2$ is equivalent to $y \circ u \subseteq H_2$ for all $u \in M$ and hence we have $(x, y) \in \Xi_{(M,H_2)}$.

2.11 Definition Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, $R \subseteq M$ non void set. We call the set R normal complex if for arbitrary $u \in M$ and arbitrary $x, y \in R$ the relation $x \circ u \subseteq R$ implies $y \circ u \subseteq R$.

2.12 Theorem Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, R subset of M. The following affirmations are equivalent:

A) R is normal complex.

B) There exists homomorphism φ of ordered semihypergroup $\mathcal{M}$ such that R

is full origin of one element for homomorphism φ.
Proof. Proposition of the theorem follows from Theorem 4.6 [10],p. 377.

2.13 Definition Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, Θ a congruence relation on $\mathcal{M}$,

$$L = \bigcup_{i \in I} H_I \subseteq M, \ H_i \in M/\Theta.$$

We say that $\{H_i\}_i \in I$ satisfies the condition **C** if there does not exist the set $J \subset I$ such that card $J > 1$ and

$$\bigcup_{i \in J} H_i$$

is normal complex in $\mathcal{M}$.

2.14 Lemma Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, Θ a congruence relation on $\mathcal{M}$,

$$L = \bigcup_{i \in I} H_I \subseteq M, \ H_i \in M/\Theta.$$

Let $\{H_i\}_i \in I$ satisfy the condition **C**. Then $H_i \in M/\Xi_{(\mathcal{M},L)}$ for every $i \in I$.
Proof. The set L is a union of classes of the congruence $\Xi_{(M,L)}$ according to 2.7. Using Lemma 2.8 we receive that $\Xi_{(M,L)}$ without the class X of the element $x \in L$ is a union of the set of classes modulo Θ from L. Since X is normal complex (see 2.12), the set of classes modulo Θ has with respect to the condition **C** just one element. Hence $\Xi_{(M,L)}$ minus the class $X \subseteq L$ blends with the class of the congruence Θ containing the element x.

2.15 Lemma Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, Θ a congruence relation on $\mathcal{M}$,

$$L = \bigcup_{i \in I} H_i \subseteq M, \ H_i \in M/\Theta.$$

and let $\{H_i\}_{i \in I}$ satisfies the condition **C**. Then $\Xi_{(M,L)} = \Xi_{(M, \cup_{i \in I} H_i)} \subseteq \Xi_{(M,H_i)}$ for every $i \in I$.
Proof. Let $(x, y) \in \Xi_{(M,L)}$. If for all $u \in M$ $x \circ u, \ y \circ u \subseteq L$ then $x \circ u, \ y \circ u$ lie in the same class H_i. This follows from 2.14 and from the stability of congruence. The sets $x \circ u$ and $y \circ u$ lapse either into the same class H_i or into complement $M - L$ of the set L for all $u \in M$ and $(x, y) \in \Xi_{(M,L)}$. Hence for al $(x, y) \in \Xi_{(M,L)}$ is also $(x, y) \in \Xi_{(M,H_i)}$ for all $i \in I$.

2.16 Theorem Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered semihypergroup, Θ a congruence relation on $\mathcal{M}$,

$$L = \bigcup_{i \in I} H_i \subseteq M, \ H_i \in M/\Theta.$$

and let $\{H_i\}_{i\in I}$ satisfies the condition **C.** Then

$$\bigcap_{i\in I} \Xi_{(M,H_i)} = \Xi_{(M,L)}$$

.

Proof. The Proposition follows from 2.9 and 2.15.

2.17 Theorem Let $\mathcal{M} = (M, \leq, \circ, 0, I)$ be an ordered hypergroupoid, Θ a congruence relation on $\mathcal{M}$, $H_1,\ H_2 \in M/\Theta$. The following propositions are equivalent:

A) $H_1 \in M/\Xi_{(M,H_2)}$ $H_2 \in M/\Xi_{(M,H_1)}$

B) $\Xi_{(M,H_1)} = \Xi_{(M,H_2)}$

C) $\Xi_{(M,H_1)} \subseteq \Xi_{(M,H_1\cup H_2)}$ and $\Xi_{(\mathcal{M},H_2)} \subseteq \Xi_{(M,H_1\cup H_2)}$

Proof. Let A) hold. Then 2.13 implies B).
We prove that B) implies C). We receive from the lemma 2.12 $\Xi_{(M,H_1)} \cap \Xi_{(M,H_2)} \subseteq \Xi_{(M,H_1\cup H_2)}$ and from the assumption B) follows $\Xi_{(M,H_1)} = \Xi_{(M,H_2)} = \Xi_{(M,H_1)} \cap \Xi_{(M,H_2)}$ and hence C) holds true.
It remains to be proven that C) implies B). We divide this par of the proof onto two parts.
α) Let $T \in M/\Xi_{(M,H_1\cup H_2)}$ and $x, y \in T$. We show that either simultaneously $(x, y) \in \Xi_{(M,H_1)}$ and $(x, y) \in \Xi_{(M,H_2)}$ or simultaneously $(x, y) \notin \Xi_{(M,H_1)}$ and $(x, y) \notin \Xi_{(M,H_2)}$. ¿From the relation $(x, y) \in \Xi_{(M,H_1\cup H_2)}$ follows that $x \circ u \subseteq H_1 \cup H_2$ is equivalent to $y \circ u \subseteq H_1 \cup H_2$ for all $u \in M$.
We call the element $u,\ u \in M$ as an element of the type one, if simultaneously $x \circ u \subseteq H_1,\ y \circ u \subseteq H_1$, or simultaneously $x \circ u \subseteq H_2,\ y \circ u \subseteq H_2$,.
We call the element $u,\ u \in M$ as an element of the type two, if simultaneously $x \circ u \not\subseteq H_1 \cup H_2,\ y \circ u \not\subseteq H_1 \cup H_2$.
We call the element $u,\ u \in M$ as an element of the type three, if either simultaneously $x \circ u \subseteq H_1,\ y \circ u \subseteq H_2$ or simultaneously $x \circ u \subseteq H_2,\ y \circ u \subseteq H_1$.
This way we distinguished all the elements $u \in M$ into three families. If no element is of the type three then simultaneously $(x, y) \in \Xi_{(M,H_1)}$ and $(x, y) \in \Xi_{(M,H_2)}$. If there exists at least one element of the type three then $(x, y) \notin \Xi_{(M,H_1)}$ and $(x, y) \notin \Xi_{(M,H_2)}$. β. If $x, y \in M$ are arbitrary elements with the property $(x, y) \in \Xi_{(M,H_1)}$ then there exists such a class $T \in \mathcal{M}/\Xi_{(M,H_1\cup H_2)}$ for which according to C) $x, y \in T$ holds true. From α follows $(x, y) \in \Xi_{(M,H_2)}$. Hence $\Xi_{(M,H_1)} \subseteq \Xi_{(M,H_2)}$ and similarly $\Xi_{(M,H_2)} \subseteq \Xi_{(M,H_1)}$. We have proved that C) implies B).
In the last part of proof we show that B) implies A). With respect to the corollary 2.10 the set H_1 is a union of $\Xi_{(M,H_1)}$ classes. Similarly H_2 is a union of $\Xi_{(M,H_2)}$ classes. From the lemma 2.11 with assumption that H_1, H_2 are $Theta$-classes we receive $H_1 \in M/\Xi_{(M,H_1)}$ and $H_2 \in M/\Xi_{(M,H_2)}$. That settles the proof.

3. Distinguishing subsets of hypergroupoids

3.1 Definition We say that the subset L of M distinguishes the hypergroupoid $\mathcal{M} = (\,M \leq, \circ, I)$ if for every pair $(x, y) \in M, x \neq y$ there exists an element $u \in M$ such that

$$x \circ u = \{s_i \mid i \in I\} \subseteq L \text{ and } y \circ u = \{t_j \mid j \in J\} \not\subset L$$
$$\text{or}$$
$$x \circ u = \{s_i \mid i \in I\} \not\subseteq L \text{ and } y \circ u = \{t_j \mid j \in J\} \subset L$$

3.2 Definition By the *depth* $d(x)$ of the element x of the partially ordered set abounded upper is meant the minimum of the set of all lengths of the chains among the elements x and I it is $min\{$ lengths of all chains $[x, I]\}$ $(d(I) = 0)$. We say that the partial ordered set M is $uniform$ when for every element $x \in M$ all chains among the elements x and I have the same length.

3.3 Definition The subset $D \subset N$ for which elements z holds $d(z) = 1$ is called the set of $dualatoms$.

3.4 Agreement Hereafter we suppose that the hypergroupoid $M = (\,M \leq, \circ, I)$ has no meet irreducible elements except for the set of dual atoms and that the ordered set M is uniform.

3.5 Theorem Let a hypergroupoid $\mathcal{M} = (\,M \leq, \circ, I)$ of the length N be given and every element $x \in M$ with $d(x) = n$ be covered by all elements $z \in M$ with $d(z) = n - 1$. Then the subset $L \subset M$ consisting from the elements $z \in M$ which have $d(z) = 0$ and $d(z) = 2\,.\,k \mid k = 1,\, 2 \ldots q$ distinguishes the hypergroupoid $\mathcal{M}$.

Proof. We prove for arbitrary pair $(x, y) \in N \times N$, $x \neq y$ the existence of an element u for which all elements of multioperation $x \circ u$ are in L and $y \circ u$ are not in L or conversely. Let $x \neq y$, $x, y \in N$.

1)Let $d(x) = d(y)$. It is sufficient to put $v = x$ end we have $x \circ x = x$ and the whole set $y \circ x$ has $d(y \circ x) = d(x) + 1$. Hence the element x is in L and the set $y \circ x$ is not in L or conversely.

2)Let $d(x)$ and $d(y)$ are both even or both odd. Without loss of generally we can suppose that both depths are even and $d(x) < d(y)$ Then there exists an element u which covers the element x such that $d(u) = d(x) + 1$ and $d(u) < d(y)$. Hence $x \circ u = u$ and $y \circ u = y$ and $d(u) = d(x) + 1$ Such the product $y \circ u$ is an element of L and $x \circ u$ is not in L.

3) Let one value of depths of elements x, y be even the second one odd. It is obvious that that $d(x) < d(y)$ or $d(y) < d(x)$. Let the first case occur. Then $x \circ x = x$ and $y \circ x = y$. With respect to the assumption in part 3 of the proof the product $x \circ x$ does not lie in L and $y \circ x$ lies in L. Such the role of u plays x.

Linking 1), 2) and 3) we prove the theorem.

3.6 Theorem Let a uniform hypergroupoid $\mathcal{M} = (\,M\ \leq, \circ, I)$ of the length N be given. Then there exists at least one distinguishing subset $L \subset M$ of $\mathcal{M}$.

Proof. We do the construction of a distinguishing subset. We begin at the element with the smallest depth it is at I and we put $L_0 = \{I\}$. The following elements are elements of the depth 1, these are dual atoms and we denote he set of all dual atoms by D. We have for every pair $(x, y) \in \{D \cup I\} \times \{D \cup I\}, x \neq y, x \neq I$ products $x \circ x = x$ and $y \circ x = I$. The role of the element u in the definition 2.2 acts the element x and $L_0 = I$ distinguishes the subset $\{D \cup I\}$. Let all pairs $(x, y) \in M \times M,\ x \neq y,\ d(x) = k - i,\ d(y) = k - j,\ i, j = 0, 1, ..., k$ are mutually distinguished by $L_0 \subset M$. Such k exists and $k \leq 1$
Let $(x, y), x \neq y, d(x) = k, d(y) = k + 1$,
1) Let us suppose that $y < x$ and that there exist u such that $u \succ\!\!- y$ and $u \parallel x$. Then $y \circ u = u$ where $d(u) = k$ and $x \circ u = \{z_q \mid q \in Q\}$ and $d(z_q) = k_0 < k$.. Every pair $(u, z_q) \mid q \in Q$ for which components their depths are smaller or equal to k is distinguished by L_0 it is there exists v such that $u \circ v \in L_0$ and simultaneously $z_q \circ v \notin L_0$ or $u \circ v \notin L_0$ and simultaneously $z_q \circ v \in L_0$ for all $q \in Q$
2) Let (x, y) be a pair of different elements for which $d(x) = k + 1,\ d(y) = k + 1$ and let there exist an element u with the property $u \succ\!\!- y$. Then $y \circ u = u$ where $d(u) = k$ and $x \circ u = \{z_q \mid q \in Q\}$ where simultaneously $d(z_q) = k_0 < k$. and we act alike as in 1.
3) Let for all doubles $(x, y),\ x \neq y,\ d(x) = d(y) - k + 1$ both elements x and y be lying under every of elements of the set of dual elements D. We denote the set of these elements as $\{t_s \mid s \in S\}$ and we define a new distinguishing subset $L_{k+1} = \{I\} \cup\ \{t_s \mid s \in S\}$ where the set S is the index set for which all indexes s the depth $d(t_s) = k + 1$. This set distinguishes all pairs (x, y) satisfying the conditions in 1, in 2 and the conditions of the part 3 of the proof. It is sufficient to take as the element u one from the elements x and y, for example x and we have $x \circ x = x$ simultaneously $x \in L_{k+1}$ and $y \circ x = \{z_q \mid q \in Q\}$ where $1 > d(z_q) \geq k$ for all indexes $q \in Q$. We see that all pairs (x, y) with the depth of their elements smaller than $k + 1$ can be distinguished by $L_{k+1} = \{I\} \cup\ \{t_s \mid s \in S\}$
4) We denote the set of elements which are covered by the elements of the set $\cup \{t_s \mid s \in S\}$ by P and we call it's elements *predecessors*. In the next steps of the construction of the distinguishing set we do the considerations obtained in the parts 1, 2 and 3 with that difference that we are working not only with the set D bat wit the set P too. In this way we can receive next sets of predecessors and the algorithm of the construction repeats with growing set of sets of predecessors.

3.7 Theorem Let a hypergroupoid $\mathcal{M} = (\,M \leq, \circ, I)$ of the length N be given. Then there exists at least one distinguishing subset $L \subset M$ of $\mathcal{M}$.

Proof. We do the construction of a distinguishing subset similarly as in the Theorem 2.8.
1) All dual atoms are distinguished by the set $L_0 = \{I\}$.
2) We suppose that all pairs $(x, y) \in M \times M$ for which $d(x) \leq k_0,\ d(y) \leq k_0$ are distinguished by L_0. Such k_0 exists and $k_0 \geq 1$ which follows from the first part of this proof.
3) Let $(x, y) \in M \times M$ be such that $d(x) \leq k_0,\ d(y) = k_0 + 1$.

a) If $x \parallel y$ then it is sufficient to put $u = x$ and we have $x \circ u = x \circ x = x$ and $y \circ u = \{z_q \mid q \in Q\}$ where $d(z_q) \le k_0$ for all $q \in Q$ and such all pairs $(x, z_q),\ q \in Q$ are according to the assumption 2 distinguished by $L_0 = \{I\}$.

b) If $x > y$ and if there exists an element $u,\ u \parallel x$ with the property $u \succ\!\!-y$ then $y \circ u = u$ and $x \circ u = \{z_q \mid q \in Q$ where $d(z_q) \le k_0$ for all indexes $q \in Q$. All pairs (u, z_q) with given properties are distinguished by the subset $L_0 = \{I\}$.

b_1) If $x > y$ and if such an $u,\ u \parallel x$ with the property $u \succ\!\!-y$ does not exist then we define a new distinguishing subset. We union all elements y satisfying b_1) with the subset $L_0 = \{I\}$ and we have $L^1{}_{k_0} = L_0 \cup \{y \mid d(y) = k_0 + 1, y < x$ for $\forall x, d(x) \le k_0\}$. It is obvious that the distinguishing of all pairs $(x, y),\ d(x) \le k_0,\ d(y) \le k_0$ will be preserved as well pairs satisfying b). We show that the subset $L^1{}_{k_0}$ distinguishes all pairs satisfying b$-$1. It is obvious that the element x does not lie in the subset L_0. The element y was assigned into the set $L^1{}_{k_0}$ Now it suffices to put $u = y$ hence $x \circ u = x \circ y = x$ and $y \circ u = y \circ y = y$ and the pair (x, y) is distinguished by $L^1{}_{k_0}$.

c) We will study all pairs (x, y) for which $d(x) \le k_0,\ d(y) = k_0 + 1$ and $x < y$. If there exists u such that $u \succ\!\!-x$ and $u \parallel y$ then $x \circ u = u$, $d(u) \le k_0$ and $y \circ u = \{z_q \mid q \in Q,\ D(z_Q) \le k_0\}$ and all pairs (u, z_q) are distinguished by L_0 according to the assumption 2 of the proof.

c_1) Let $d(x) \le k_0,\ d(y) = k_0 + 1$ and $x < y$. We suppose now, that such u as in c) does not exist. We define a new distinguishing subset. We union all elements y satisfying c_1 with $L^1{}_{k_0}$ and we receive $L_{k_0+1} = L^1{}_{k_0} \cup \{y \mid d(y) = k_0 + 1,\ x > y$ for $\forall x,\ d(x) \le k_0\}$. All pairs $(x, y) \in M \times M$ satisfying all conditions till c_1 are distinguished by $L^1{}_{k_0}$. Let $(x, y) \in M \times M$ satisfies c_1. It is sufficient to put x as u and hence $x \circ u = x \circ x = x$ and $y \circ u = y \circ x = y$. Such $x \notin L_{k_0+1}$ simultaneously $y \in L_{k_0+1}$ and the pair $(x, y) \in M \times M$ is distinguished too.
The set of all pairs created from the set of all predecessors of the subset L_{k_0+1} can be by this subset distinguished. We repeat for the following construction of the distinguishing subset the algorithm aliquot to that one which is described in steps 2 and 3 of this proof with only one difference that we suppose in the step 2 all pairs $(x, y) \in M \times M$ for which $d(x) \le k_1, d(y) \le k_1$ where $k_1 \ge k_0 + 1$. The set M is finite hence after the finite steps of applications of given algorithm we receive the distinguishing subset of the hypergroupoid $\mathcal{M}$.

3.8 Remark In the following fourth paragraph there will be as example 1. partial ordered semihypergroup given. Hereafter some congruences will be shown and also an example of distinguishing subsets will be created.

4. Examples

4.1 Example Let M be an underlying ordered set of the commutative hypergroupoid $\mathcal{M}$ given by Fig.2, The hyperoperation $\circ$ is given in the Table 1 .

4.2 Example Let M be an underlying partial ordered set of the commutative hypergroupoid $\mathcal{M}$ given by Fig.2. The hyperoperation $\circ$ is given in the Table 1.

Let $L_1 = \{0, a, c, e\}$ be a subset of M. Then $\Xi_{(\mathcal{M},L_1)}$ is a congruence relation (see 2.5).
Let $x, y \in L_1$then for $u \in L_1$ $x \circ u$, $y \circ u \in L_1$ and for $u \in M - L_1 = L_2$ $x \circ u$, $y \circ u \notin L_1$
Let $x, y \in L_2$ then for all $u \in M$ $x \circ u$, $y \circ u \notin L_1$.
The system of classes of the congruence relation $\Xi_{(\mathcal{M},L_2)}$ is the same.

4.3 Example Let M be an underlying partial ordered set of the commutative hypergroupoid $\mathcal{M}$ given by Fig.2. The hyperoperation $\circ$ is given in the Table 1.
We define a subsets $L_1 \subseteq M$ such that $L = \{g, h, i, j, k, l\}$ and $L_2 = \{0, a, b, c, d, e, f, I\}$ then L_1 and L_2 distinguish the commutative hypergroupoid $\mathcal{M}$.

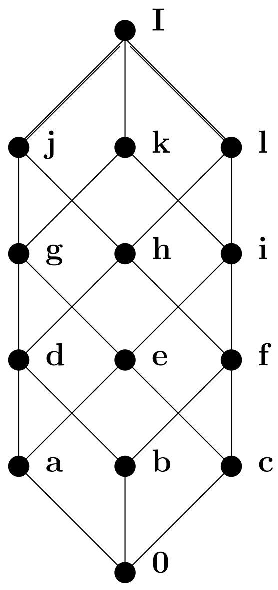

Figure 2

	0	a	b	c	d	e	f	g	h	i	j	k	l	1
0	0	a	b	c	d	e	f	g	h	i	j	k	l	1
a	a	a	d,i	e	d	e	i,j	g	h	i	j	k	l	1
b	b	d,i	b	f,g	d	g,i	f	g	h	i	j	k	l	1
c	c	e	f,g	c	g,l	e	f	g	h	i	j	k	l	1
d	d	d	d	g,l	d	g,l	h	g	h	l,k	j	k	l	1
e	e	e	g,i	l	g,l	l	i,j	g	j,l	i	j	k	l	1
f	f	i,j	f	f	h	i,j	f	j,k	h	i	j	k	l	1
g	g	g	g	g	g	g	j,k	g	j	k	j	k	I	I
h	h	h	h	h	h	j,l	h	j	h	I	j	k	I	I
i	i	i	i	i	l,k	i	i	k	I	i	I	k	I	I
j	j	j	j	j	j	j	j	j	j	I	j	I	I	I
k	k	k	k	k	k	k	k	k	k	k	I	k	I	I
l	l	l	l	l	l	l	l	I	l	l	I	I	l	I
1	I	I	I	I	I	I	I	I	I	I	I	I	I	I

Table 1 - multiplication ∘ on the hypergroupoid from 4.1.

Proof of 4.3. We prove the assertion of the example such that we define for every pair $(x, y) \in M \times M$ an element $u \in M$ for which the condition from 3.1 is satisfied. We do it with the aid of tables. The doubles and their distinguishing elements are given in the following tables.

x,y	u	x∘u	y∘u
0,a	f	e	h,i
0,b	e	e	g,i
0,c	d	d	h
0,d	f	f	h
0,e	d	d	g,l
0,f	d	d	h
0,g	l	l	I
0,h	k	k	I
0,i	j	j	I
0,j	k	k	I
0,k	j	j	I
0,l	j	j	I
0,I	j	j	I

x,y	u	x∘u	y∘u
a,b	f	h,i	f
a,c	d	d	h,g
a,d	e	e	g,l
a,e	d	d	g,l
a,f	d	d	h
a,g	e	e	g
a,h	k	k	I
a,i	j	j	I
a,j	k	k	I
a,k	j	j	I
a,l	j	j	I
a,I	j	j	I

x,y	u	x∘u	y∘u
b,c	b	b	f
b,d	b	b	d
b,e	c	f,g	e
b,f	b	b	f
b,g	b	b	g
b,h	k	k	I
b,i	j	j	I
b,j	k	k	I
b,k	j	j	I
b,l	j	j	I
b,I	d	d	I

x,y	u	x∘u	y∘u
c,d	c	c	g,l
c,e	h	h	j,l
c,f	c	c	f
c,g	c	c	g
c,h	e	e	j,l
c,i	c	c	i
c,j	c	c	j
c,k	c	c	k
c,l	c	c	l
c,I	f	f	I

x,y	u	x∘u	y∘u
d,e	e	l	g,l
d,f	d	d	h
d,g	l	l	I
d,h	d	d	h
d,i	j	j	I
d,j	l	l	I
d,k	j	j	I
d,l	j	j	I
d,I	j	d	I

x,y	u	x∘u	y∘u
e,f	e	e	i,j
e,g	e	e	g
e,h	h	j,l	h
e,i	j	j	I
e,j	k	k	I
e,k	j	j	I
e,l	j	j	I
e,I	e	e	I

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