Requirements for Solution of Linear Programming

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Abstract: This paper shows a method for solution of optimizing problems which is other than the usually used *Simplex method.* The simplex method is considered one of the basic models from which many linear programming techniques are directly and indirectly derived. The simplex method is an iterative process which approaches, step by step, an optimum solution in such a way that an objective function of maximization or minimization is fully reached. Each iteration in this process consists of shortening the distance (mathematically and also graphically) from the objective function to the intercepted vertex of a convex set determined by the inequalities that describe the problem. The simplex method is not the only technique known and used for solving linear programming problems. Other methods are more useful for the pedagogical expediency, see e.g. R. Dorfman, P.A.Samuelson, and R.M.Solov, *Linear Programming and Economic Analysis*, New. York: McGraw-Hill Book Comp. Inc., 1958. I introduce another method than simplex method. This method is be based on the principle of graphical method of optimization of linear problems for two variables, but my method is generalized for n variables and arbitrary finite number of inequalities describing the problem.

Key words. Linear programming, system of inequalities, disposal and slack variable, dummy variable, objective function, iteration, key row, key column, key element, polyhedron, hyperplane.

1. The leading article

1.1 Introduction The general problem of linear programming is usually formulated as follows:
Let $a_{ij},\ b_i,\ c_j\ (i = 1, 2, \dots, m\ ;\ j = 1, 2, \dots, n)$ be given real numbers and let us denote $I_1 \subset I = \{i = 1, 2, \dots, m\}$ and $J_1 \subset J = j = 1, 2, \dots, n\}$. The problem of maximizing of the function

$$\sum_{i=1}^{n} c_j\, x_j \qquad (1)$$

on the set of

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_j \qquad (i \in I_1) \qquad (2)$$

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_j \qquad (i \in I - I_1) \qquad (3)$$

$$x_j \geq 0 \qquad (j \in J_1) \qquad (4)$$

is called maximizing problem of linear programming in mixed form if $I_1 \neq \emptyset$, $I_1 \neq I$ or $J_1 \neq J$.

The problem of linear programming given by (1) till (4), where $I_1 = I$ and $J_1 = J$, is the problem of maximizing of the function

$$\sum_{j=1}^{n} c_j x_j \qquad (5)$$

on the set of linear independent system of linear inequalities

$$\sum_{j=1}^{n} a_{ij}\, x_j \leq b_j \qquad (i \in I_1) \qquad (6)$$

$$x_j \geq 0 \qquad (j \in J_1) \qquad (7)$$

is called maximizing problem of linear programming in the form of inequalities.

With respect to the fact that for arbitrary set $M \subset R^n$ where R^n is n dimensional vector space and for arbitrary linear function $z : M \rightarrow R^n$

$$\min z(\mathbf{x}) = \text{-}\ \max(-z(\mathbf{x})), \text{ where } \mathbf{x} \in R^n$$

holds then if one of extremes exists we can transform also the minimizing problem into the problem with linear equations or linear inequalities. We do the rearrangement by multiplication by number -1.

2. The solution of general problem

We desist from the condition (4) and hence also from (7) in the following considerations. We rewrite the system (6) and add the objective function as the last row into the form:

$$\begin{array}{l} a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n + b_1 \geq 0 \\ a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{2n}\, x_n + b_2 \geq 0 \\ \ldots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \ldots + a_{mn}\, x_n + b_m \geq 0 \\ c_1\, x_1 + c_2\, x_2 + \ldots + c_n\, x_n = 0 \end{array} \qquad (8)$$

We call the set $\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \subset R^n$ of elements a polyhedron. We call the polyhedron opened if $m \leq n$. We know that the objective function $z(\mathbf{x})$, $\mathbf{x} \in R^n$ receives its optimal values at vertices or at all areas of polyhedron. In the first part of the computational procedure we find one of the polyhedron vertices and we transform it into the coordinate origin simultaneously. Simultaneously we transform all hyperplanes of the polyhedron and the objective function with respect to the given transformation. We continue as follows:

We select arbitrary hyperplane and we denote it by the index $i \in \{1, 2, \ldots, m\}$. We divide the whole column at the variable x_1 by coefficient a_{i1} and we put simultaneously

$$x_1 = x_1' - a_{i2}\,x_2 - a_{i3}\,x_3 \ldots - a_{in}\,x_n - b_i. \qquad (9)$$

We introduce the transformation relation (9) into the system (8) and the mathematical representation of the problem transforms for $m > 1$ into the following form:

$$\begin{array}{l}
\frac{a_{11}}{a_{i1}}\,x_1' + (a_{12} - \frac{a_{11}\,a_{i2}}{a_{i1}})\,x_2 + (a_{13} - \frac{a_{11}\,a_{i3}}{a_{i1}})\,x_3 + \ldots + (a_{1n} - \frac{a_{11}\,a_{in}}{a_{i1}})\,x_n + b_1 - b_i\,\frac{a_{11}}{a_{i1}} \geq 0 \\
\frac{a_{21}}{a_{i1}}\,x_1' + (a_{22} - \frac{a_{21}\,a_{i2}}{a_{i1}})\,x_2 + (a_{23} - \frac{a_{21}\,a_{i3}}{a_{i1}})\,x_3 + \ldots + (a_{2n} - \frac{a_{21}\,a_{in}}{a_{i1}})\,x_n + b_2 - b_i\,\frac{a_{21}}{a_{i1}} \geq 0 \\
\quad . \\
\frac{a_{i-1,1}}{a_{i1}}\,x_1' + (a_{i-1,2} - \frac{a_{i-1,1}\,a_{i2}}{a_{i1}})\,x_2 + (a_{i-1,3} - \frac{a_{i-1,1}\,a_{i3}}{a_{i1}})\,x_3 + \ldots + (a_{i-1,n} - \frac{a_{i-1,1}\,a_{in}}{a_{i1}})\,x_n + b_{i-1} - b_i\,\frac{a_{i-1,1}}{a_{i1}} \geq 0 \\
x_1' \quad + \quad 0 \quad + \quad 0 + \ldots + \quad 0 + \quad 0 \quad \geq 0 \\
\frac{a_{i+1,1}}{a_{i1}}\,x_1' + (a_{i+1,2} - \frac{a_{i+1,1}\,a_{i2}}{a_{i1}})\,x_2 + (a_{i+1,3} - \frac{a_{i+1,1}\,a_{i3}}{a_{i1}})\,x_3 + \ldots + (a_{i+1,n} - \frac{a_{i+1,1}\,a_{in}}{a_{i1}})\,x_n + b_{i+1} - b_i\,\frac{a_{i+1,1}}{a_{i1}} \geq 0 \\
\quad . \\
\frac{a_{21}}{a_{i1}}\,x_1' + (a_{22} - \frac{a_{21}\,a_{i2}}{a_{i1}})\,x_2 + (a_{23} - \frac{a_{21}\,a_{i3}}{a_{i1}})\,x_3 + \ldots + (a_{2n} - \frac{a_{21}\,a_{in}}{a_{i1}})\,x_n + b_2 - b_i\,\frac{a_{21}}{a_{i1}} \geq 0
\end{array}$$

In the following step we choose some rows where the coefficient at the variable x_2 is different from zero arbitrary. The existence of such road follows from the assumption that m linear rows are independent. Further we suppose that this assumption is satisfied by the row $s, s \leq m$. It is obvious that $s \neq i$. We continue in such a way that we divide the whole second column by the expression

$$a_{s2} - \frac{a_{si}\,a_{i2}}{a_{i1}}$$

and then we introduce the following transformation:

$$x_2 = -\frac{a_{s1}}{ai1} + x_2' - (a_{s3} - \frac{a_{s1}\,a_{i3}}{a_{i1}})\,x_3 + \ldots - (a_{s,n} - \frac{a_{s,1}\,a_{in}}{a_{i1}})\,x_n - b_s + b_i\,\frac{a_{s,1}}{a_{i1}}.$$

After this transformation the s-th row will be of the form:

$$0 \quad x_2' + \quad 0 \quad + \quad 0 + \ldots + \quad 0 + \quad 0 \quad \geq 0$$

We continue till we do all the $m < n$ transformations by analogy. Thus we calculate one point of one edge of polyhedron which transformed into the coordinate origin.
In the account that $m \geq n$ we find after n transformations one vertex of polyhedron which transformed into the coordinate origin. The whole calculation is done on computer therefore we calculate only with the matrix of coefficients of polyhedron. We apply all the steps of transformation to the objective function

$$c_1\,x_1 + c_2\,x_2 + \ldots + c_n\,x_n + 0 \geq 0$$

and we receive after the first transformation

$$\frac{c_1}{a_{i1}}\,x_1' + (c_2 - \frac{c_1\,a_{i2}}{a_{i1}})\,x_2 + (c_3 - \frac{c_1\,a_{i3}}{a_{i1}})\,x_3 + \ldots + (c_n - \frac{c_1\,a_{in}}{a_{i1}})\,x_n + 0 - b_i\,\frac{c_1}{a_{i1}} \geq 0$$

Further adaptations of coefficients of the objective function run over simultaneously with the adaptations of coefficients of polyhedron such as it was given in the previous description of the hash algorithm applied onto polyhedron.

As the next step we extend the matrix of coefficients describing the system (8) with the objective function which is of the type $(m+1)\times(n+1)$ such that we add the matrix of the type $n\times(n+1)$ which consists of unit matrix type $n\times n$ with added column vector of zeros of the length n. This step is necessary for explicit expression of the point of edge optionally vertex of polyhedron and the optimal value of the objective function. All the above described affinite transformation are applied onto such expanded matrix. After the above described transformation algorithm we obtain the original coordinates of the point of edge or the vertex of polyhedron which is transformed into the coordinate origin in the last column of the matrix $n\times(n+1)$. We show the expanded matrix of coefficients of the type $(m+1+n)\times(n+1)$ before the transformation algorithm.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ . & . & . & \dots & . & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \\ c_1 & c_2 & c_3 & \dots & c_n & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

3. Optimizing and decision making process

We suppose that $m\geq n$ and that the transformation algorithm transformed one of the polyhedron vertices were transformed into the coordinate origin. If after this transformation all the coefficients in the first m rows of the last $(n+1)$-the column are nonnegative numbers and simultaneously all transformed coefficients of the objective function it is $c_k^{'}$ in the $(m+1)$-th row negative the maximizing process of the objective function $z(\mathbf{x})$ is finished. In the last n rows of the $(n+1)$-the column there are original coordinates of the polyhedron vertex in which the objective function acquires its maximum and the value of this maximum is at the position $[m+1,\, n+1]$ of transformed matrix.
If previous situation does not occur then it is necessary to do the following analysis. We suppose for coefficients in the first m rows of the $(n+1)$-th column nonnegative again but some of transformed coefficients of the objective function in the $(m+1)$-th row is positive. Let this situation in the j_0-th column occur. We look at all transformed coefficients in the j_0-th column. If all transformed coefficients of the polyhedron $a_{ij_0}^{'}$ are nonnegative then the problem does not have any solution. It is possible to get along this edge incident to polyhedron to infinity. The polyhedron is not bounded and the solution does not exist.

4. Example

4.1 Remark I drafted a program for explanation of the given method which address is on the server Pal of the Technical University in Brno: Q: \vyuka\ matemat\Tomsova\Polyhedron\ matice. exe. We do an application of this program. Data are denoted as CONCRETE

4.2 Example Maximize the objective function $z = 3x + 5y$ in the area bounded by the following restrictions:

1) $x \geq 0$
2) $y \geq 0$
3) $x \leq 5$
4) $x + 2y \leq 12$
5) $2x + 3y \leq 19$

Solution: We line a figure for the better graphical preview where the area of polyhedron will be bounded with bisectors suitable to the restrictions with the vertices $0 = [0,\ 0]$, $A = [5,\ 0]$, $B = [5,\ 3]$, $C = [2,\ 5]$, $D = [0,\ 6]$

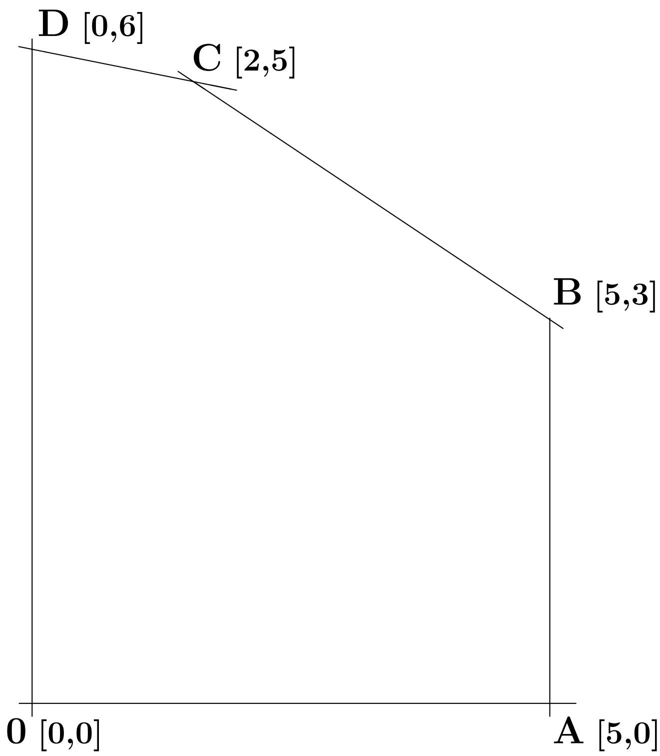

Figure 1

After start-up of program we obtain the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 5 \\ -1 & -2 & 12 \\ -2 & -3 & 19 \\ 3 & 5 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The first five rows represent the bound of the given polyhedron. The sixth row contains the objective function and in the last two rows there is a unit matrix. The program continues with given translations given in the previous part of article and the output block has the following form:

$$\begin{pmatrix} 3 & -2 & 2 \\ -2 & 1 & 5 \\ -3 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 31 \\ 3 & -2 & 2 \\ -2 & 1 & 5 \end{pmatrix}.$$

The sixth element in the last column is the greatest value of the objective function and the last elements in the two last rows are the coordinates of the point in which the greatest value of the objective function arrives. We see that the problem has just one solution and it is at the point C and the value of the objective function is 31. Such a simple example was chosen for demonstration. Program solves more complicated problems and responds to the problem of the existence of a solution.

References

[1] CHURCHMAN, CH.W., ACKOFF R.L., ARNOFF L. : *Introduction to Operation Research*. New York, John Wiley & Sons. Inc. 1957.

[2] KLAPKA J., DVOŘÁK J., POPELA P.: *Metody operačního výzkumu.* Brno, VUTIUM, 2001

[3] RAIS K.: *Vybrané kapitoly z operační analýzy.* Brno PGS 1985.

[4] VACULÍK J., ZAPLETAL J.: *Podpůrné metody rozhodovacích procesů* . Masarykova univerzita, Brno 1998.

[5] WALTER J. a kol.: *Operační výzkum.* Praha SNTL,1973.

[6] ZAPLETAL J.: *Operační analýza.* Kunovice, Skriptorium VOŠ 1995