# Fuss' Problem of the Chord-Tangent Quadrilateral

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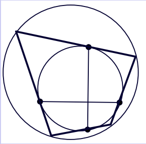

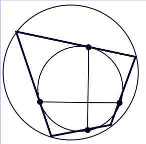

# References

[1 ] Dörrie, H. *100 Great Problems of Elementary Mathematics, Their History and Solution.* Dover Publications Inc., New York. 1965

[2 ] Weisstein, E. W. *Bicentric Quadrilateral.* MathWorld – A Wolfram Web Resource. URL <http://mathworld.wolfram.com/BicentricQuadrilateral.html>

[3 ] O'Connor, J. J., Robertson, E. F. *Nicolaus Fuss.* School of Mathematics and Statistics University of St Andrews, Scotland. URL <http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Fuss.html>

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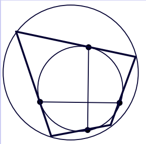
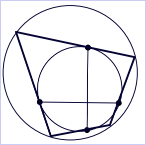
# Nicolaus Fuss

## (1755-1826)

* Swiss mathematician
* 1772 went to St Petersburg in Russia
* worked under L. Euler's direction
* solved many problems on spherical geometry, trigonometry, differential geometry, differential equations ...
* was interested in bicentric polygons

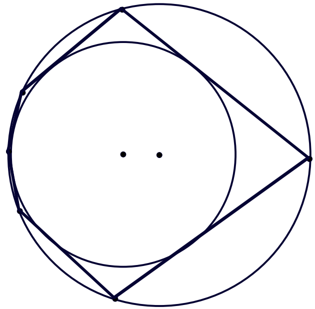






# Bicentric polygon

(or a chord-tangent polygon) is a polygon that is inscribed in one circle and circumscribed about another.



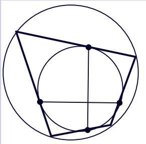
# Fuss' Problem of the Chord-Tangent Quadrilateral

*To find the relation between the radii and the line joining the centers of the circles of circumscription and inscription of a bicentric quadrilateral.*

N. Fuss found a solution for

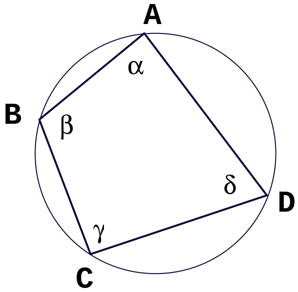
tetragon, pentagon, heptagon, hexagon, and octagon.





# Chordal quadrilateral

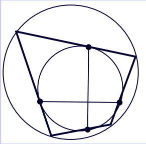
(or cyclic quadrilateral) is a quadrilateral admitting circumcircle. The sum of its opposite internal angles is the straight angle.

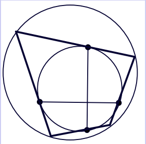



$$\alpha + \gamma = 180^\circ$$

$$\beta + \delta = 180^\circ$$

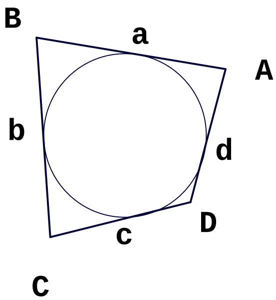






# Tangential quadrilateral

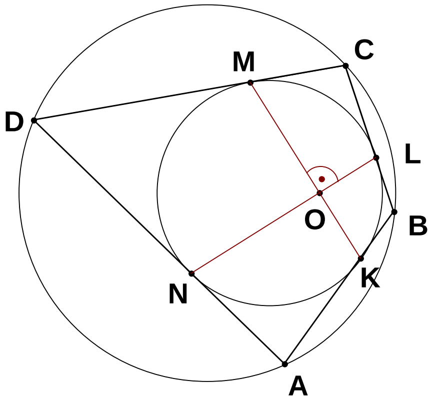
is a quadrilateral admitting incircle. The sum of its opposite sides equals the sum of the other opposite sides.



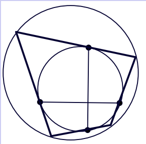

$$a + c = b + d$$

# Bicentric quadrilateral

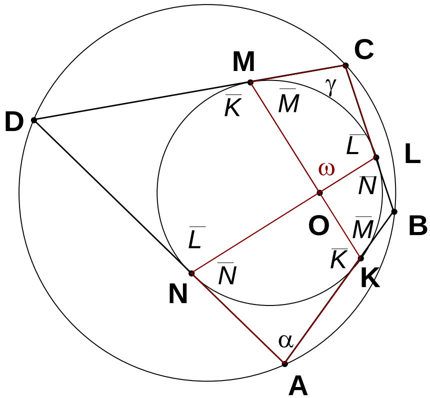
The tangency chords of the two pair of opposite sides are perpendicular to each other.





# Proof



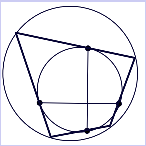

$$\gamma + \bar{M} + \bar{L} + \omega = 360^\circ$$

$$\alpha + \bar{K} + \bar{N} + \omega = 360^\circ$$

$$\underbrace{\alpha + \gamma}_{180^\circ} + \underbrace{\bar{M} + \bar{K}}_{180^\circ} + \underbrace{\bar{L} + \bar{N}}_{180^\circ} + 2\omega = 720^\circ$$

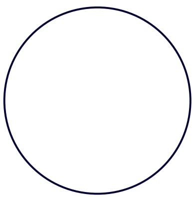
$$\Rightarrow \quad \omega = 90^\circ$$



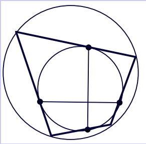


# Construction of the bicentric quadrilateral

* circle,

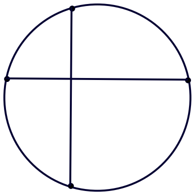




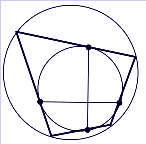


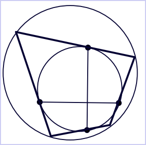
# Construction of the bicentric quadrilateral

* circle,
* two perpendicular chords,







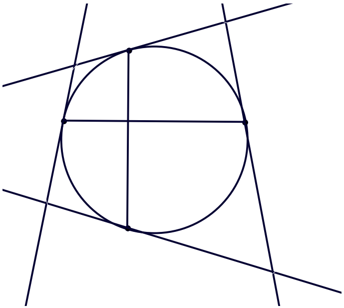


# Construction of the bicentric quadrilateral

∗ circle,

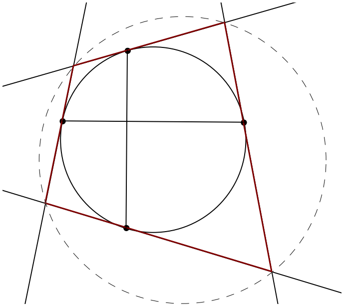
∗ two perpendicular chords,

∗ tangents at endpoints of chords,

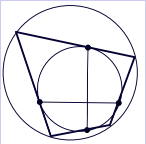


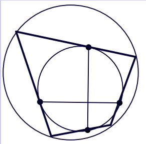
# Construction of the bicentric quadrilateral

∗ circle,
∗ two perpendicular chords,
∗ tangents at endpoints of chords,
∗ points of intersection of tangents
form vertices of the desired quadrilateral.



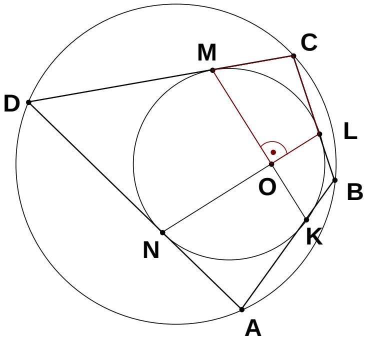


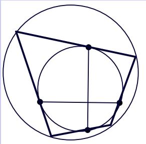




The tangency chords of the two pairs of opposite sides divide every bicentric quadrilateral into four quadrilaterals with these properties:

- ∗ the right angle is at the common vertex,
- ∗ adjacent vertices lie on the circle,
- ∗ sides not containing the common vertex are tangents to this circle.

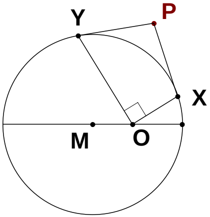




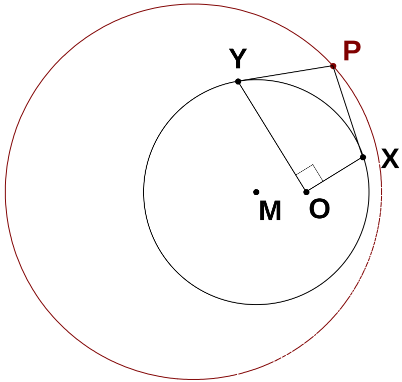
# The locus

Examining quadrilaterals mentioned above by investigating locus of points P (see following) leads to the solution of the Fuss's problem.

* Let $k$ be a circle, $O$ be the point inside,
* $X$, $Y$ be the points on $k$ giving right angle $XOY$,
* $P$ be the point of intersection of tangents touching $k$ at $X$ and $Y$.

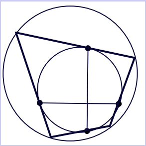


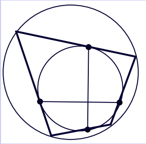
# Visualization of the locus




Is it a circle?

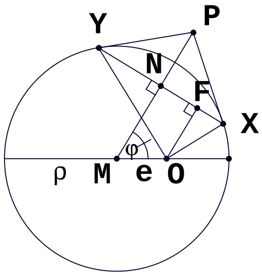






# Proof

Let $e = |MO|$,
$p = |MP|$,
$\varphi = |\angle OMP|$.




* $\triangle OXY$ is right-angled
  $\Rightarrow |OF|^2 = |FX| \cdot |FY|$
* $\overleftrightarrow{MP}$ is a bisector of $XY$
  $\Rightarrow |NX| = |NY|$, $|\angle MNX| = 90^\circ$

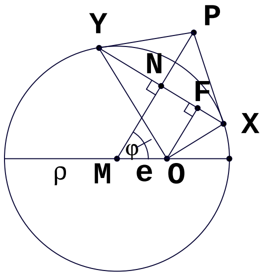
$$|OF| = |MN| - e \cdot \cos\varphi$$

$$|FX| = |NX| - e \cdot \sin\varphi$$

$$|FY| = |NX| + e \cdot \sin\varphi$$

$*\ \triangle MXN$ is right-angled

$$\Rightarrow |NX|^2 = \rho^2 - |MN|^2$$




$*\ \triangle MXP$ is right-angled

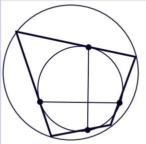
$$\Rightarrow \rho^2 = p \cdot |MN|$$

Substituting for $|MN|$ and arranging:

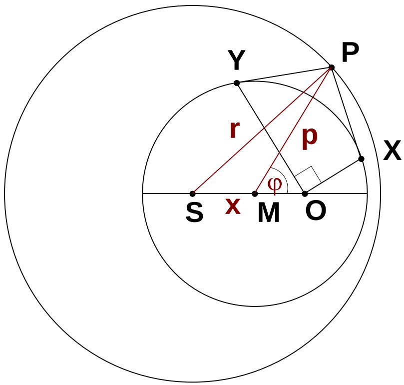
$$\frac{2\rho^4}{\rho^2 - e^2} = 2\frac{\rho^2 e}{\rho^2 - e^2} p \cos\varphi + p^2$$

$p$ and $\varphi$ are variables dependent on the position of point $P$; $\rho$ and $e$ are invariant for given circle $k$ and point $O$.





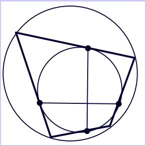
Let's assume that P lies on a circle with the radius $r$ and the centre S $\in \overleftrightarrow{MO}$.




Cosine law in $\triangle SMP$ ($x = |SM|$, $r = |SP|$):

$$r^2 = x^2 + p^2 + 2xp \cos\varphi$$





Comparing last two equations:

$$\frac{2\rho^4}{\rho^2 - e^2} = 2\frac{\rho^2 e}{\rho^2 - e^2} p\cos\varphi + p^2$$

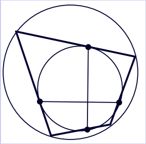
$$r^2 - x^2 = 2xp\cos\varphi + p^2$$

($p$ and $\varphi$ depend on the position of point $P$; $\rho$ and $e$ are constants.)

For $x = \frac{\rho^2 e}{\rho^2 - e^2}$, the radius $r$ is constant.

$\Rightarrow$ it is a circle.





Eliminating $e$ from following equations

$$\frac{2\rho^4}{\rho^2 - e^2} = r^2 - x^2$$

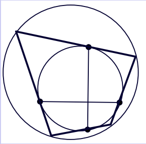
$$\frac{\rho^2 e}{\rho^2 - e^2} = x$$

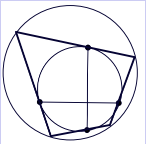
leads to the relation among

* the radius of given circle$\rho$,
* the radius of found circle $r$
* and the distance of their centers $x$:

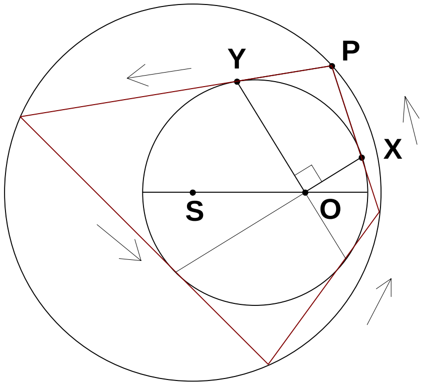
$$2\rho^2(r^2 + x^2) = (r^2 - x^2)^2$$







Given two circles, one inside another. Starting in $P$ (see figure) we get the bicentric quadrilateral.




The position of $P$ on the outer circle is arbitrary, moving $P$ we obtain all bicentric quadrilaterals with given incircle and circumcircle.

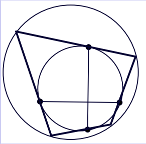
# Conclusion

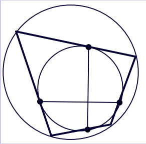
Foregoing procedure leads not only to proving the locus is a circle but also to the (more important) solution of the Fuss' problem:

*Given $\rho$ and $r$, the radii of incircle and circumcircle, the distance $x$ of their centers satisfies the equation*

$$2\rho^2(r^2 + x^2) = (r^2 - x^2)^2.$$







## Solution analysis

Under the given conditions, $x \in (0, r - \rho)$

- $r \geq \rho\sqrt{2} \quad \Rightarrow \quad$ one solution
  $x = \sqrt{r^2 + \rho^2 - \rho\sqrt{4r^2 + \rho^2}}.$
- $r < \rho\sqrt{2}$ no solution
  (no quadrilateral can be constructed).

Derived equation can be modified to:

$$\frac{1}{(r - x)^2} + \frac{1}{(r + x)^2} = \frac{1}{\rho^2}.$$