Fuss' Problem of the Chord-Tangent Quadrilateral (Abstract)

Mgr. Barbora Stastna
Masaryk University, Faculty of Science
e-mail: stastna@mail.muni.cz

Nicolaus Fuss (1755-1826) was Swiss mathematician. He spent most of his life in St. Petersburg in Russia, because he was recommended by Daniel Bernoulli for the post of Leonhard Euler's secretary. Working at the St. Petersburg Academy he wrote papers in spherical geometry, differential geometry, differential equations and many other topics.

Apart from other things N. Fuss investigated bicentric polygons, figures admitting both incircle and circumcircle, in other words both tangential and chordal. He found the relation between the radii and the line joining the centers of these circles for bicentric tetragon, pentagon, hexagon, heptagon and octagon.

Fuss' problem

To find the relation between the radii and the line joining the centers of the circles of circumscription and inscription of a bicentric quadrilateral.

Bicentric quadrilateral

The tangency chords of the two pairs of opposite sides of a bicentric quadrilateral are perpendicular to each other (see figure 1).

Conversely, every cyclic quadrilateral where the tangency chords of the two pairs of opposite sides are perpendicular to each other is bicentric. Consequently starting with two perpendicular chords of a circle, constructing tangents in their extremities, the bicentric quadrilateral is created.

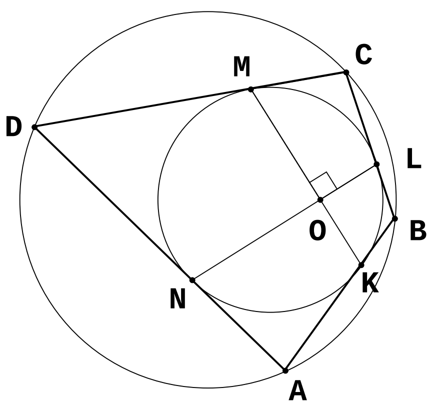


Figure 1: Bicentric quadrilateral

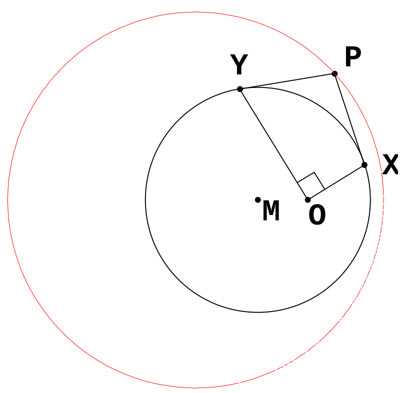
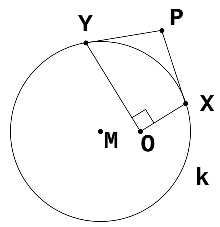


Figure 2: Visualization of the locus

The locus

Lines KM and LN divide the bicentric quadrilateral $ABCD$ in four quadrilaterals $AKON$, $BLOK$, $CMOL$ and $DNOM$ with some common properties. Examining this sort of quadrilateral $OXPY$ by investigating locus of points P is conductive:



- Let k be a circle, O be the point inside,
- X, Y be the points on k giving right angle XOY,
- P be the point of intersection of tangents touching k in X and Y.

Visualizing the locus (see figure 2) the locus seems to be a circle. The proof leads not only to proving the locus is a circle but above all to the solution of the Fuss' problem.

Conclusion

Given ρ and r the radii of incircle and circumcircle, then the distance x of their centers satisfies the equation

$$2\rho^2(r^2 + x^2) = (r^2 - x^2)^2.$$

For $x \in (0, r-\rho)$, $r \geq \rho\sqrt{2}$ this equation has one solution

$$x = \sqrt{r^2 + \rho^2 - \rho\sqrt{4r^2 + \rho^2}}.$$

For $r < \rho\sqrt{2}$ it has no solution and no bicentric quadrilateral can be constructed.

Given two circles, one inside another, satisfying the equation above. Starting in an arbitrary point P on the outer circle, drawing a tangent to the inner circle, from the point of intersection with the outer circle a tangent to inner one again and so on, we return to P and get the bicentric quadrilateral. Moving P we obtain all bicentric quadrilaterals for given incircle and circumcircle.

Derived equation can be arranged:

$$\frac{1}{(r-x)^2} + \frac{1}{(r+x)^2} = \frac{1}{\rho^2}.$$

References

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