A note to an s-map

Olga Nánásiová*

Slovak University of Technology, Bratislava

Abstract

The modeling of "quantum probability", in another words, the study of random events, which are not measurable simultaneously, provides us with interesting results. In this paper we summarising some results from the study of conditionnig non compatible random events.

1 Introduction and basic notion

Conditional probability plays a basic role in the classical probability theory. Some of the most important areas of the theory such as martingales, stochastic processes rely heavily of this concept. Conditional probabilities on a classical measurable space are studied in several different ways, but result in equivalent theories. The classical probability theory does not decsribe the causality model.

The situation charges when non-standard spaces are considered. For example, it is a well known that the set of random events in quantum mechanics experiments is a more general structure than Boolean algebra. In the quantum

*Supported by grant VEGA1/0273/03,

logic approach the set of random events is assumed to be a quantum logic (orthomodular lattice L). Such model we can find not only in the quantum theory, but for example, in the economics, biology etc. We will show such such a simple situation in the Example 1.

In this paper we will study a conditional state on a quantum logic using Renyi's approach (or Bayesian principle). This approach helps us to define independence of events and differently from the situation in the classical theory of probability, if an event a is independent of an event b, then the event b can be dependent on the event a (problem of causality) ([13], [**?**]). We will show that we can define a s-map (function for simultaneous measurements on a quantum logic). It can be shown that if we have the conditional state we can define the s-map and conversely. By using the s-map we can introduce joint distribution also for noncompatible observables on a quantum logic. Moreover, if x is an obsevable on L and B is Boolean sub-algebra of L, we can construct an observable $z = E(x|B)$, which is a version of conditional expectation of x but it need not to be necessarily compatible with x.

Definition 1. 1 *Let L be a nonempty set endowed with a partial ordering $\leq$. Let there exists the greatest element* 1 *and the smallest element* 0*. We consider operations supremum* ($\vee$)*, infimum* $\wedge$ *(the lattice operations) and an map* $\perp$: $L \rightarrow L$ *defined as follows.*

(i) For any $\{a_n\}_{n\in\mathcal{A}} \in L$*, where* $\mathcal{A} \subset \mathcal{N}$ *is finite,*

$$\bigvee_{n\in\mathcal{A}} a_n, \bigwedge_{n\in\mathcal{A}} a_n \in L.$$

(ii) For any $a \in L$ $(a^\perp)^\perp = a$.

(iii) If $a \in L$*, then* $a \vee a^\perp = 1$.

(iv) If $a, b \in L$ *such that* $a \leq b$*, then* $b^\perp \leq a^\perp$.

(v) If $a, b \in L$ such that $a \leq b$ then $b = a \vee (a^{\perp} \wedge b)$ (orthomodular law).

Then $(L, 0, 1, \vee, \wedge, \perp)$ is said to be the orthomodular lattice (briefly OML).

Let L be an OML. Then elements $a, b \in L$ will be called:

- *orthogonal* ($a \perp b$) iff $a \leq b^{\perp}$;
- *compatible* ($a \leftrightarrow b$) iff there exist mutually orthogonal elements $a_1, b_1, c \in L$ such that

$$a = a_1 \vee c \quad \text{and} \quad b = b_1 \vee c.$$

Definition 1. 2 *A map $m : L \to [0, 1]$ such that*

(i) $m(0) = 0$ and $m(1) = 1$.

(ii) If $a \perp b$ then $m(a \vee b) = m(a) + m(b)$

is called a state on L.

Definition 1. 3 *[13] Let L be an OML. A subset $L_0 \subset L - \{0\}$ is called a conditional system (CS) in L if the following conditions hold:*

- *If $a, b \in L_0$, then $a \vee b \in L_0$.*
- *If $a, b \in L_0$ and $a < b$, then $a^{\perp} \wedge b \in L_0$.*

Let $A \subset L$. Then $L_0(A)$ is the smallest CS, that contains the set A.

Definition 1. 4 *[13] Let L be an OML and let L_0 be a CS in L. Let $f : L \times L_0 \to [0, 1]$. If the function f fulfills the following conditions:*

(C1) for each $a \in L_0$ $f(., a)$ is a state on L;

(C2) for each $a \in L_0$ $f(a, a) = 1$;

(C3) if $\{a_i\}_{i=1}^n \in L_0$ *and* a_i *are mutually orthogonal, then for each* $b \in L$

$$f(b, \bigvee_{i=1}^{n} a_i) = \sum_{i=1}^{n} f(a_i, \bigvee_{i=1}^{n} a_i) f(b, a_i);$$

then it is called a conditional state.

Definition 1. 5 *[13] Let* L *be an OML and* f *be a conditional state. Let* $b \in L$, $a, c \in L_0$ *such that* $f(c, a) = 1$. *Then* b *is independent of* a *with respect to the state* $f(., c)$ *(*$b \asymp_{f(.,c)} a$*) if* $f(b, c) = f(b, a)$.

If L_0 be CS and $f : L \times L_0 \to [0, 1]$ is a conditional state, then ([13])

(i) Let $a^\perp, a, c \in L_0$, $b \in L$ and $f(c, a) = f(c, a^\perp) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $b \asymp_{f(.,c)} a^\perp$.

(ii) Let $a, c \in L_0$, $b \in L$ and $f(c, a) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $b^\perp \asymp_{f(.,c)} a$.

(iii) Let $a, c, b \in L_0$, $b \leftrightarrow a$ and $f(c, a) = f(c, b) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $a \asymp_{f(.,c)} b$.

2 An s-map

Definition 1. 6 *Let* L *be an OML. The map* $p : L^n \to [0, 1]$ *will be called s-map if the following conditions are met:*

(s1) $p(1, ..., 1) = 1$*;*

(s2) if there exist i, j*, such that* $a_i \perp a_j$*, then* $p(a_1, ..., a_n) = 0$*;*

(s3) if $a_i \perp b_i$*, then*

$$p(a_1, ..., a_i \vee b_i, ..., a_n) = p(a_1, ..., a_i, ..., a_n) + p(a_1, ..., b_i, ..., a_n),$$

for $i = 1, ..., n$.

Definition 3. 1 *Let x be an obsevable on L and B be a Boolean sub-algebra of L and f be conditional state on L such that $L_c = L - \{0\}$. Then the observable z will be called a conditional expectation of x with respect to B in the state $f(.,1)$ iff for any $b \in B - \{0\}$*

$$f(x,b) = f(z,b).$$

We will denote $z := E_f(x|B)$.

It is clear that if L be a Boolean algebra, then $E_f(x|B)$ is known the conditional expectation.

Proposition 2. 1 *Let L be an OML and let p be an s-map. Then*

(1) if $a_i \perp a_j$, then $p(a_1, ..., a_n) = 0$;

(2) for any $a \in L$, a map $\nu : L \to [0,1]$, such that $\nu(a) := p(a, ..., a)$ is a state on L;

(3) for any $(a_1, ..., a_n) \in L^n$ $p(a_1, ..., a_n) \leq \nu(a_i)$ for each $i = 1, ..., n$;

(4) if $a_i \leftrightarrow a_j$, then

$$p(a_1, ..., a_n) = p(a_1, ..., a_{i-1}, a_i \wedge a_j, ..., a_j \wedge a_i, a_{j+1}, ..., a_n).$$

Let $\bar{a} = (a_1, ..., a_n) \in L^n$. Let us denote $\pi(\bar{a})$ a permutation of $(a_1, ..., a_n)$.

Proposition 2. 2 *Let L be an OML. Let p be an s-map and let $(a_1, ..., a_n) \in L^n$.*

(1) If there exists $i \in \{1, ..., n\}$, such that $a_i = 1$, then

$$p(a_1, ..., a_n) = p(a_1, ..., a_{i-1}.a_j, a_{i+1}, ..., a_n)$$

for each $j = 1, ..., n$.

(2) If there exist $i \neq j$ such that $a_i = a_j$, then

$$p(a_1, ..., a_n) = p(\pi(a_1, ..., a_n)).$$

(3) If there exist i, j such that $a_i \leftrightarrow a_j$, then

$$p(a_1, ..., a_n) = p(\pi(a_1, ..., a_n)).$$

Let $\Pi(\bar{a})$ be the set of all permutions and let

$$\bar{a}^{(i)}_{(k)} = (a_1, ..., a_{k-1}, a_k, a_{k+1}, ..., a_{i-1}, a_k, a_{i+1}, ..., a_n).$$

Corollary 2. 1 *Let L be an OML. Let p be an s-map and let $\bar{a} \in L^n$.*

(1) If there exists $i \in \{1, ..., n\}$, such that $a_i = 1$, then

$$p(\bar{a}) = p(\bar{b})$$

for each $\bar{b} \in \bigcup_k \Pi(\bar{a}^{(i)}_{(k)})$.

(2) If there exist $i \neq j$ such that $a_i = a_j$, then

$$p(\bar{a}) = p(\bar{b})$$

for each $\bar{b} \in \bigcup_k \Pi(\bar{a}^{(i)}_{(k)})$.

(3) If there exist i, j such that $a_i \leftrightarrow a_j$, then

$$p(\bar{a}) = p(\bar{b})$$

for each $\bar{b} \in \bigcup_k \Pi(\bar{a}^{(i)}_{(k)})$.

Example 2. 1 *Let $n = 3$ and $a, b \in L$. If $\bar{a} = (a, a, b)$, then $\Pi(\bar{a}) = \{(a, a, b), (b, a, a), (a, b, a)\}$ and $\bar{a}^{(1)}_{(3)} = (b, a, b)$, $\bar{a}^{(2)}_{(3)} = (a, b, b)$,$\bar{a}^{(1)}_{(2)} = (a, a, b)$. Hence*

$$p(a, a, b) = p(a, b, a) = p(b, a, a) = p(b, b, a) = p(a, b, b) = p(b, a, b).$$

Let $n = 4$ and $a, b, c \in L$. If $\bar{a} = (a, b, c, c)$, then $\bar{a}^{(4)}_{(2)} = (a, b, c, b)$ and

$$p(a, a, b, c) = p(a, b, c, a) = p(b, b, c, a) = ... = p(c, a, b, c).$$

Definition 3. 2 *Let L be an OML and let p be an s-map. If $x_1, ..., x_2$ are observables on L, then the map*

$$p_{x_1,...,x_n} : \mathcal{B}(R)^n \to [0, 1],$$

such that

$$p_{x_1,...,x_n}(E_1, ..., E_n) = p(x_1(E_1), ..., x_n(E_n))$$

is called the joint distribution of the observables $x_1, ..., x_n$.

Definition 3. 3 *Let L be an OML and let p be an s-map. If $x_1, ..., x_2$ be observables on L, then the map*

$$F_{x_1,...,x_n} : R^n \to [0, 1],$$

such that

$$F_{x_1,...,x_n}(r_1, ..., r_n) = p(x_1(-\infty, r_1), ..., x_n(-\infty, r_n))$$

is called the joint distribution function of the observables $x_1, ..., x_n$.

Definition 3. 4 *Let L be an OML and let p be an s-map. If $x_1, ..., x_2$ be observables on L, then a marginal distribution function is*

$$\lim_{x_i \to \infty} F_{x_1,...,x_i,...,x_n}(r_1, ..., r_i, ..., r_n).$$

Definition 3. 5 *Let L be an OML and let p be an s-map. Let $x_1, ..., x_2$ be observables on L and $F_{x_1,...,x_n}$ be the joint distribution function of the observables $x_1, ..., x_n$. Then we say, that $F_{x_1,...,x_n}$ has the property of commutativity if for each $(r_1, ..., r_n) \in R^n$*

$$F_{x_1,...,x_n}(r_1, ..., r_n) = F_{\pi(x_1,...,x_n)}(\pi(r_1, ..., r_n)).$$

It is clear that $F_{x_1,...,x_n}$ has the property of commutativity if and only if

$$p(x_1(E_1), ..., x_n(E_n)) = p(\pi(x_1(E_1), ..., x_n(E_n))),$$

for each $E_i \in \mathcal{B}(R)$, $i = 1, ..., n$.

Proposition 3. 1 *Let L be an OML and let p be an s-map. Let $x_1, ..., x_2 \in \mathcal{O}$ and let $F_{x_1,...,x_n}(r_1, ..., r_n)$ be the joint distribution function of the observables $x_1, ..., x_n$.*

(1) For each $(r_1, ..., r_n) \in R^n$ $0 \leq F_{x_1,...,x_n}(r_1, ..., r_n) \leq 1$;

(2) If $r_i \leq s_i$, then $F_{x_1,...,x_n}(r_1, ..., r_i, ..., r_n) \leq F_{x_1,...,x_n}(r_1, ..., s_i, ..., r_n)$.

(3) For each $i = 1, ..., n$

$$\lim_{r_i \to \infty} F_{x_1,...,x_n}(r_1, ..., r_n) = F_{x_1,...,x_n}(r_1, ...r_{i-1}, 1, r_{i+1}, ..., r_n).$$

(4) For each $i = 1, ..., n$

$$\lim_{r_i \to -\infty} F_{x_1,...,x_n}(r_1, ..., r_n) = 0.$$

(5) If there exist i, j, such that $i \neq j$ and $x_i \leftrightarrow x_j$, then

$$F_{x_1,...,x_n}(r_1, ..., r_n) = F_{\pi(x_1,...,x_n)}(\pi(r_1, ..., r_n)).$$

Proposition 3. 2 *Let L be an OML and let p be an s-map. Let $x_1, ..., x_n \in \mathcal{O}$ and let $F_{x_1,...,x_n}(r_1, ..., r_n)$ be the joint distribution function. Compatibility of just two observables imply the total commutativity.*

Proposition 3. 3 *Let L be an OML and let $x_1, ..., x_n \in \mathcal{O}$. Then there exist a probability space $(\Omega, \Im, P)$ and random variables $\xi_1, ..., \xi_n$ on it, such that*

$$F_{x_1,...,x_n}(r_1, ..., r_n) = F_{\xi_1,...,\xi_n}(r_1, ..., r_n)$$

and P_{ξ_i} such that

$$P_{\xi_i}((-\infty, r)) = \nu(x_i(-\infty, r)),$$

where $r \in R$ and $i = 1, ..., n$ is the probability distribution of the random varaible ξ_i, If we consider a quantum model as an OML, a marginal distribution function defined by using an s-map has the property of commutativity. It follows that, in general, it need not true that that

$$F_{x_1,...,x_n}(t_1,...,t_n) = F_{x_1,...,x_{n+1}}(t_1,...,t_n,\infty),$$

where $F_{x_1,...,x_n}(t_1,...,t_n)$, $F_{x_1,...,x_{n+1}}(t_1,...,t_n,\infty)$ are joint distribution funtions and $x_1,...,x_{n+1}$ are observables on L. Consequently, we can find such an s-map and an s_{n+1}-map such that

$$p(a_1,...,a_n) \neq p(a_1,...,a_n,1)$$

on L. Moreover if

$$p(a_1,...,a_n) = p(a_1,...,a_n,1)$$

on L, then the s-map has the property of commutativity. This is not true in general, either ([16]).

References

[1] a1 Accardi L., *The probabilistic roots of the quantum mechanical paradoxis. The wawe-particle dualism. A tribute to Louis de Broglie on the 90th Birthday*, ed. S. Diner, G. Lochak and F. Selleri (D. Reidel Publ. Company, Dortrecht, 297-330, 1984).

[2] Cox R., *The algebra of probable inference*, J. Hopkins Univ. Press, Baltimore (1961)

[3] Dvurečenskij A., Pulmannová S., New Trends in Quantum Structures, Kluwer Acad. Publ., (2000).

[4] Gudder S.,P., Quantum probability spaces, AMS, vol 21 (1969), 296-302.

[5] Guz W., Conditional probability and the axiomatic structure of quantum mechanics, Fortsch. Phys., vol 29 (1982) 345-379.

[6] Khrennikov A. Yu., Contextual viewpoint to quantum stochastics, *J. Math. Phys.*, **44**, N. 6, 2471-2478 (2003).

[7] Yu A., Khrenikov A., Origin of quantum probabilities. Proc. Conf. "Foundations of Probability and Physics", Vaxjo-2000, Sweden; p. 180-200,

[8] Khrennikov A. Yu., *Interpretations of Probability*, VSP Int. Sc. Publ., Utrecht (1999).

[9] Khrennikov A. Yu., *Linear representations of probabilistic transformations induces by context transitions*, J. Phys. A: Math. Gen., **34** 9965-9981 (2001).

[10] Khrennikov A. Yu., *Hyperbolic quantum mechanics*, quant-ph/0101002 (2000).

[11] Khrennikov A. Yu., *Contextual viewpoint to Greenberger-Horne-Zeilinger paradox*, Phys. Lett., A, **278**, 307-314 (2001). WSP,Singapore (2001).

[12] Kolmogoroff, A. N. Grundbegriffe der Wahrscheikchkeitsrechnung, Springer, Berlin, (1933).

[13] Nánásiová O., On conditional probabilities on quantum logic., Int. Jour. of Theor. Phys., vol 25, (1987), 155 - 162.

[14] O. Nánásiová, A. Yu. Khrennikov, Observables on a quantum logic. *Foundation of Probability and Physics-2,* Ser. Math. Modelling in Phys., Engin., and Cogn. Sc., vol. 5, 417-430, Växjö Univ. Press, (2002).

[15] O. Nánásiová, Map for Simultaneus Measurements for a Quantum Logic, Int. Journ. of Theor. Phys., Vol. 42, No. 8, 1889-1903, (2003).

[16] Khrenikov A., Nánásiová O., Representation theorem of observables on a quantum system, Preprint (2003).

[17] Nánásiová O., *On conditional probabilities on quantum logic*, Int. Jour. of Theor. Phys., **25**, 155 - 162 (1987).

[18] Pták P., Pulmannová S., Quantum Logics, Kluwer Acad. Press, Bratislava (1991).

[19] Riečan B., Neubrun T., Measure theory, Kluwer Acad. Press, Bratislava.

[20] Varadarajan V., Geometry of quantum theory, Princeton, New Jersey, D. Van Nostrand, (1968)

Department of Mathematics and Descriptive Geometry,

Faculty of Civil Engineering,

Slovak University of Technology, Bratislava

Radlinského 11,

813 68 Bratislava, Slovakia

e-mail: olga@math.sk olga.nanasiova@stuba.sk