Qudrature of a parabola

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1 Introduction

Quadrature of a parabola is one of the most famous problems of antiquity. The greatest name connected with this problem is that of Archimedes. He was the first who found the solution of this problem and proved it. It was about 240 B.C. In this presentation let's say a little about Archimedes' solution and then show how to prove Archimedes' theorem by a focus definition of a parabola.

At first, let me explain how to do a quadrature of some plane object. It is a problem of constructing a square of the same area as a given plane object using only a ruler and a pair of compasses. Quadrature of a parabola is then the problem of finding and constructing the square of the same area as a parabolic segment with a common base.

By a parabolic segment we understand a plane object, bounded by the arch of parabola with end points A, B and chord AB (we call it base). Let's define a vertex of parabola as a point where the tangent to the parabola, parallel to the base, is touching the parabola p.

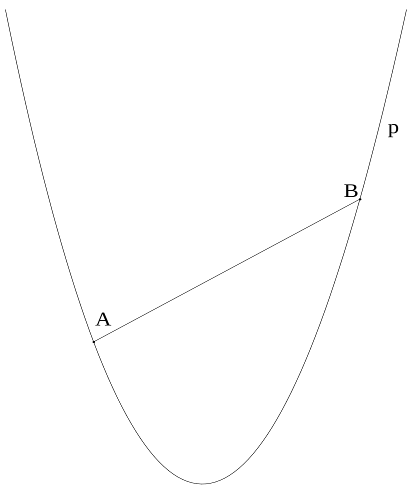

Fig. 1: segment of a parabola with the base AB

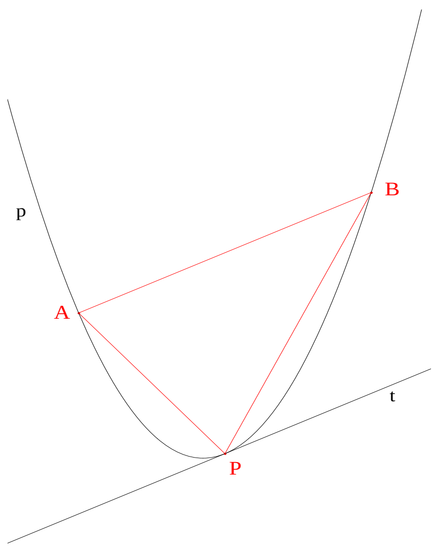

Fig. 2 : vertex of a segment of parabola

2 Archimedes' theorem

Archimedes started this theorem:
The area of every segment of a parabola means four-thirds the area of a triangle with the same base AB and vertex P as the segment.

$$\mathbf{S} = \frac{\mathbf{4}}{\mathbf{3}}\mathbf{S}_{\triangle \mathbf{ABP}}.$$

This is sufficient for doing the quadrature of parabola, because if we have the mentioned triangle, we can easily construct the rectangle of the same area. Then we can use any of the Euclid theorems to obtain a square of the same area.

Archimedes proved this theorem twice - by physical and mathematical means. His physical solution is based on the stability on a lever.

Archimedes' proofs are based on three theorems, which were not proved by him. Let's show how we can prove two of them by the focus definition of parabola and show that Archimedes' theorem can be completely concluded from them. We will show the procedure which is not Archimedes' own but the main idea is the same as his.

The theorems are:
Proposition 1
If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PS, and if AB be a chord parallel to the tangent to the parabola at P and meeting PS in S, then

$$AS = SB.$$

Conversely, if $AS = SB$, the chord AB will be parallel to the tangent at P.

Proposition 2
If in a parabola AB be a chord parallel to the tangent at P, and if a straight line be drawn through P which is either itself the axis or parallel to the axis, and which meets AB in S and the tangent at A to the parabola in C, then

$$PS = PC.$$

(see [1])

3 Focus definition of a parabola

We know that parabola is a set of points which have the same distance from the directrix and from the focus. We mark the point which is the perpendicular projection of the point (A) to directrix by a letter with a low index 0

(A_0). It results from the definition that if we know the focus of the parabola and the point on directrix A_0, we can find the relevant point of parabola A as a point of intersection of perpendicular line through the point of directrix A_0 and perpendicular bisector of abscissa FA. Perpendicular bisector of FA is equally tangent of the parabola, what can be shown in this picture:

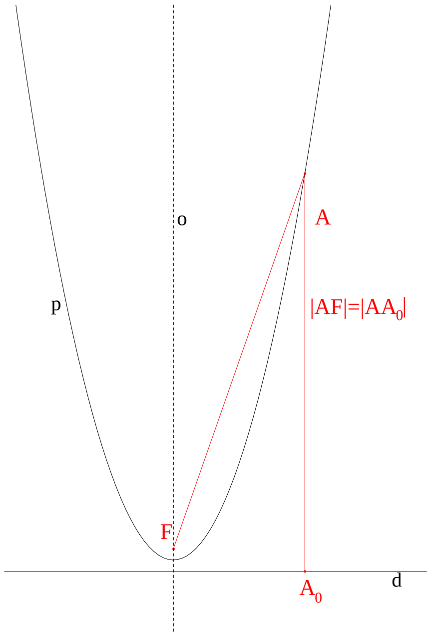

Fig. 3: focus definition of a parabola

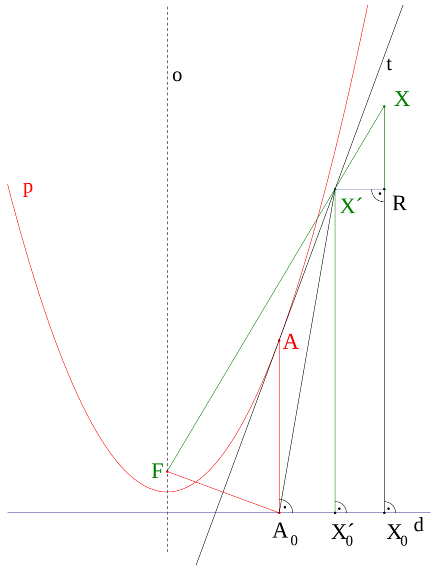

Fig. 4: tangent to a parabola

We can prove it:

$$| FX |=| FX´ | + | XX´ |,$$

simultaneously:

$$| FX´ |=| A_0X´ |\geq| X_0X´ | \qquad | XX´ |>| RX | .$$

By addition we obtain:

$$| FX |>| X_0´X´ | + | RX |=| X_0R | + | RX |=| X_0X | .$$

4 Archimedes' triangle

Let's introduce one more definition:

Let C be the point of intersection of two tangents at different points A, B of parabola. The triangle ABC is Archimedes' triangle drawn

to a segment of parabola with the base AB.

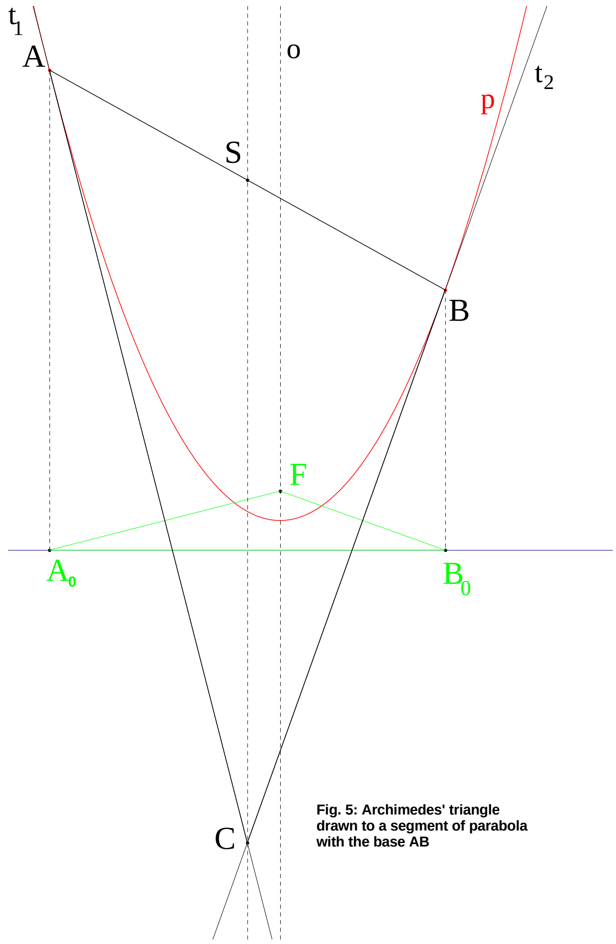

Fig. 5: Archimedes' triangle drawn to a segment of parabola with the base AB

Let’s show its geometrical properties:
If we have a triangle A_0B_0F, then AC is a perpendicular bisector of A_0F and BC is a perpendicular bisector of B_0F. C, the point of their intersection, must be also the point of a third perpendicular bisector of triangle A_0B_0F. This bisector is also the midline of trapezoid AA_0B_0B.

So we have proved this theorem:

The middle of the base AB of segment of parabola and vertex C of Archimedes’ triangle lie on a line, which is parallel to the axis. (1)

In other words:

The median to the base of the Archimedes' triangle is parallel to the axis.

Let's mark the point of intersection of median CS and the parabola p as P. Let the point of intersection of tangent to parabola at P and AC (BC) be A_1 (B_1). We have to show that the point P is the vertex of a segment of parabola with the base AB (we must prove that A_1B_1 is parallel to AB) and is, at the same time, the middle of the median CS.

Let's consider the triangle APA_1. It is the Archimedes' triangle drawn to a segment of parabola with the base AP. If we mark the middle of the base as G, then according to a theorem (1) above, A_1G is parallel to the axis of parabola. A_1G is also the midline of triangle ACP. Therefore the point A_1 is the middle of AC.

The same sequence of reasonings we may accomplish for the triangle BPB_1. So B_1 is the middle of BC. Then A_1B_1 is the midline of the triangle ABC which means that our two conditions (P is a middle of median CS and A_1B_1 is parallel to AB) are fulfilled.

We have proved the theorem:

The vertex P of a segment of a parabola with the base AB is a middle of midline CS of Archimedes' triangle ABC. (2)

Theorems (1), (2) are another formulation of Archimedes' propositions 1, 2 mentioned above.

5 Division of the Archimedes' triangle

In the picture we can see that the tangent A_1B_1 and chords AP, BP divide Archimedes' triangle ABC drawn to a segment of parabola with the base AB (let's mark it triangle of 1st level) into four triangles:

- internal triangle APB bounded by the chords AB, AP and BP
- external triangle A_1CB_1 bounded by the tangents at points A, B and P
- two residual triangles drawn to the segments with the bases AP and BP, which are also Archimedes' triangles (of 2nd level)

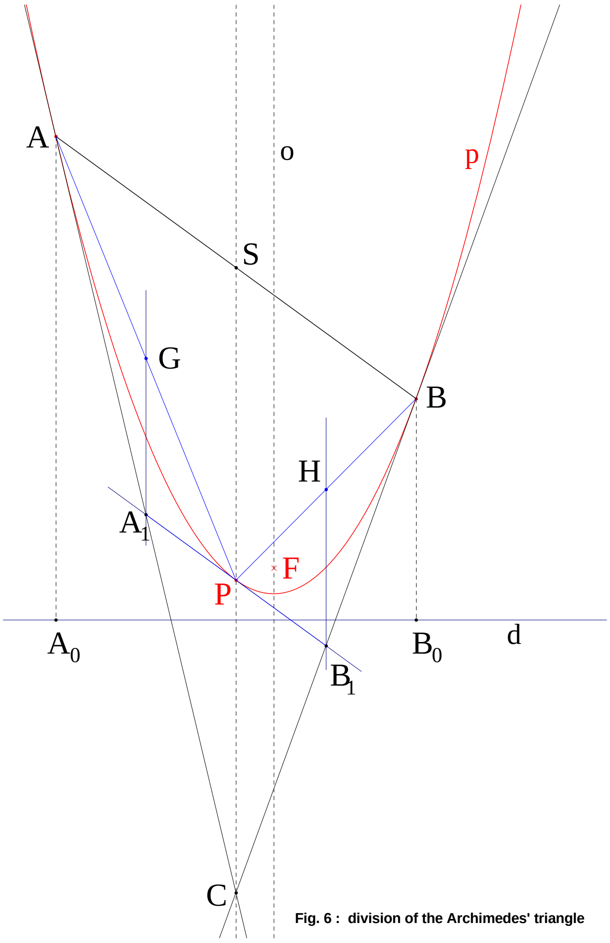

Fig. 6 : division of the Archimedes' triangle

For every residual Archimedes' triangle we can make the division again. We get four other Archimedes' triangles of 3rd level. We can repeat this process infinitely and by this we can obtain the geometrical sequence of numbers of Archimedes' triangles of different levels. There is one internal triangle corresponding with every Archimedes' triangle. Simultaneously we get the same numbers of internal triangles of each level. So we have also the geometrical sequence of numbers of internal triangles of different levels. These internal triangles will fill up the segment of parabola.

If we mark the area of the Archimedes' triangle ABC as S, then the area of corresponding internal triangle is equal to $\frac{S}{2}$, because these triangles have the same base AB and ratio of heights is $2:1$ (P lies on the midline of the triangle). In the same ratio there are the areas of internal and external triangle (they have the same height, the base A_1B_1 is a half of the base AB). The residual triangles have the same areas. They have the same base (P bisects midline A_1B_1) and the same height (chord AB is parallel to tangent at P). Every residual triangle has an area of $\frac{S}{8}$.

6 Area of a segment of parabola

If we repeat this consideration, we get geometrical sequence

$$S, \frac{S}{8}, \frac{S}{8^2}, \ldots$$

for the areas of Archimedes' triangles. We obtain geometrical sequence

$$\frac{S}{2}, \frac{S}{2.8}, \ldots.$$

for the areas of the internal triangles. Now, it is necessary to add the geometrical series of the areas of internal triangles (every area is multiplied by the number of triangles of each level):

$$\frac{S}{2} + 2.\frac{S}{2.8} + 4.\frac{S}{2.8^2} + \ldots$$

It is not difficult, we get

$$\frac{S}{2} + 2.\frac{S}{2.8} + 4.\frac{S}{2.8^2} + \cdots = \frac{\frac{S}{2}}{1 - 2.\frac{1}{8}} = \frac{2S}{3}$$

for the sum.
The Archimedes' theorem is therefore proved.

Reference

[1] The works of Archimedes, Edited by T.L.HEATH, Dover Publ. Inc., New York, 2002

[2] Heinrich Dörrie, 100 Great Problems of Elementary Mathematics, Their History and Solution, Dover Publ., New York, 1965

[3] Jindřich Bečvář, Ivan Štoll, Archimedes, největší vědec starověku, Prometheus, Praha, 2005