Application of The Finite Element Method to the Geodetic Boundary Value Problem

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Abstract

This presentation deals with the external geodetic boundary value problem for the determination of the heigh anomaly. We apply Finite Element Method to this problem with Neumann and Dirichlet boundary conditions and compare the solution with the solution of the Boundary Element Method.

Keywords: Geodetic boundary value problem, Finite Element Method.

1 Formulation of the Geodetic Boundary Value Problem

The basic quantity describing the gravity field is the **gravity potential** $W(\mathbf{x})$ that consists of the gravitational potential $V_g(\mathbf{x})$ generated by the Earth and the centrifugal potential $V_c(\mathbf{x})$ arose from spinning of the Earth. In applications, it is also used an idealized (normal) model of the Earth rotating with the same angular velocity as the Earth. Its surface potential is equal to the potential on geoid and its mass is the same as the mass of the Earth. Then the generated potential is called the **normal potential** and denoted $U(\mathbf{x})$. The difference between the actual gravity potential and this normal potential is called the **disturbing potential** $T(\mathbf{x})$. Neglecting the atmosphere, the disturbing potential is a harmonic function outside the Earth i.e., it satisfies the Laplace equation $\Delta T(\mathbf{x}) = 0$. The development of satellite technologies have brought new opportunities in geodesy. With gravimetric measurements we can get the normal $\gamma(\mathbf{x})$ and actual $g(\mathbf{x})$ gravity at the same point i.e., also the **gravity disturbance** $\delta g(\mathbf{x}) = g(\mathbf{x}) - \gamma(\mathbf{x}), \mathbf{x} \in R^3$. The same quantity can be obtained by applying the gradient operator to the definition of the disturbing potential $\nabla T(\mathbf{x}) = \nabla W(\mathbf{x}) - \nabla U(\mathbf{x}) = g(\mathbf{x}) - \gamma(\mathbf{x}), \mathbf{x} \in R^3$. Although the theoretical problem deals with infinite domain, for the Finite Element Method we constructed an artificial boundary. If the artificial boundary has radius large enough, we can consider Dirichlet boundary conditions in the form $T(\mathbf{x}) = 0$. Then the geodetic boundary value problem can be defined:

$$
\begin{aligned}
& \Delta T(\mathbf{x}) = 0, \quad \mathbf{x} \in R^3 - \Omega, \\
\Gamma_1 \ :\ & \nabla T(\mathbf{x}) = -\delta g(\mathbf{x}) \quad \text{at} \quad |\mathbf{x}| = R_1, \\
\Gamma_2 \ :\ & T(\mathbf{x}) = 0, \quad \text{at} \quad |\mathbf{x}| = R_2,
\end{aligned} \tag{1}
$$

where Γ_1 is a boundary of the Earth and Γ_2 is the artificial boundary. We note that in our first implementation we use sphere as the Earth approximation.

2 Solution of the Geodetic Boundary Value Problem

The Finite Element Method (FEM) is a numerical method for solving PDEs. It assumes discretization of the domain by a set of subdomains called the finite elements. The derivation of element equations is given through constructing the *weak form* of the differential equation, namely

$$\int_{\Omega^e} \nabla w \nabla T dxdydz - \int_{\Gamma^e} w q_n d\sigma = 0, \tag{2}$$

where w is a weight function, Ω^e is an finite element and Γ^e is an element boundary. Then we approximate the solution by polynomials over Ω^e, e.g. for 8-nodes 3D element holds: $T^e(x, y, z) = c_1 + c_2x + c_3y + c_4z + c_5xy + c_6xz + c_7yz + c_8xyz$. On the other hand the approximate polynomial is expressed as a linear combination of an interpolation functions Ψ_j^e i.e., $T(x, y, z) \approx T^e(x, y, z) = \sum_{j=1}^{8} t_j^e \Psi_j^e(x, y, z)$, where t_j^e denoted the value of T^e at j-th node. The next step is the *creating of the finite element model* by substituting the approximate solution into the weak form and considering $w = \Psi_i, i = 1...8$

$$\sum_{j=1}^{8} t_j \int_{\Omega^e} \nabla \Psi_i^e \nabla \Psi_j^e dxdydz - \int_{\Gamma_1^e} \Psi_i q_n d\sigma = 0, \tag{3}$$

so in the matrix form: $\sum_{j=1}^{8} K_{ij}^e t_j - Q_i^e = 0$, where K_{ij}^e is an element stiffness matrix and Q_i^e are "fluxes" through element faces. The final step is creating of the global model using continuity of numerical solution and balance of fluxes on interelement boundaries.

3 Numerical results

Our computational domain is the space above the sphere of radius $R_1 = 6371km$ where Neumann boundary conditions are presented up to the sphere with radius $R_2 = 20000km$ where Dirichlet boundary conditions are considered. For numerical experiments we use software ANSYS with 8-nodes elements and compared our solution with the solution by the Boundary Element Method.

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