Finite Volume Schemes for Tensor Diffusion in Image Processing

Olga Drblíková
Slovak University of Technology in Bratislava,
Faculty of Civil Engineering,
Department of Mathematics and Descriptive Geometry
e-mail: drblikov@math.sk

Image processing has a high status in modern world, since many scientific subdisciplines collect and use image data. Due to acquisition, storage, transmission and display, the images can degrade. In order to produce images of higher quality, the mathematicians and computer scientists try to find mathematical models and numerical schemes which provide it.

One of these models is also Weickert's model which is given as

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) &= 0, \qquad \text{in } Q_T \equiv I \times \Omega, & (1)\\ u(x,0) &= u_0(x), \quad \text{in } \Omega, & (2)\\ < D\nabla u, n > &= 0, \qquad \text{on } I \times \partial\Omega, & (3) \end{aligned}$$

where u denotes an unknown function $u(x,t)$, $D \equiv D(u)$ denotes the diffusion tensor, Ω is a two dimensional rectangular image domain and I is a scaling (time) interval. This matrix depends on the eigenvalues and on the eigenvectors of the structure ternsor $J = \nabla u (\nabla u)^T$. The eigenvalues of the diffusion tensor are choosen such that the smoothing is strong in one direction and low in the perpendicular direction. Due to this fact, application of the model (1)-(3) is very suitable in any situation, where the improvement of structure coherence is diserable.

Our numerical scheme is derived with the help of finite volume method. We provide a semi-implicit in time discretization and use a divergence theorem to get

$$\frac{u_K^n - u_K^{n-1}}{k} - \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \phi_\sigma m(\sigma) \quad = \quad 0, \tag{4}$$

with

$$\phi_\sigma \approx \frac{1}{m(\sigma)} \int_\sigma D_\sigma^{n-1} \nabla u^n \cdot \mathbf{n}_{K,\sigma} ds, \tag{5}$$

where K is a finite volume, σ denotes an edge of K, u_K^n is a value of u for K in $n-th$ time step, k is a time step, $m(K)$ is a measure of K, $m(\sigma)$ is a measure of σ, ε_K is a set of all edges $\sigma \subset \partial K$ and $\mathbf{n}_{K,\sigma}$ is the normal unit vector to σ outward to K.

One possibility how to derive an approximation ϕ_σ of the flux is obtained with the help of co-volume mesh. The specific name (diamond-cell) of this method (see [1]) is due to the choice of co-volume as a diamond-shaped polygon. The co-volume χ_σ associated to σ is constructed around each edge by joining four co-volume vertices (i.e. endpoints of this edge denoted by x_N and x_S and midpoints of finite volumes x_E and x_W which are common to this edge). The co-volume edges are denoted by $\bar{\sigma}$.

In order to have an approximation (5) of the diffusion flux on the edge σ, we first derive, using divergence theorem, an approximation of the averaged gradient on χ_σ, which is denoted by p_σ,

$$p_\sigma = \frac{1}{m(\chi_\sigma)} \int_{\chi_\sigma} \nabla u dx = \frac{1}{m(\chi_\sigma)} \int_{\partial\chi_\sigma} u \mathbf{n}_{\chi_\sigma,\bar\sigma} ds \tag{6}$$

and further

$$p_\sigma = \frac{1}{m(\chi_\sigma)} \sum_{\bar\sigma\in\partial\chi_\sigma} \frac{1}{2}\left(u_{N_1(\bar\sigma)} + u_{N_2(\bar\sigma)}\right) m(\bar\sigma)\mathbf{n}_{\chi_\sigma,\bar\sigma}, \tag{7}$$

where $N_1(\bar\sigma)$ and $N_2(\bar\sigma)$ are the endpoints of an edge $\bar\sigma \subset \partial\chi_\sigma$ and $\mathbf{n}_{\chi_\sigma,\bar\sigma}$ is the normal unit vector to $\bar\sigma$ outward to χ_σ. The values at the centres x_E and x_W are given by finite volume values u_E and u_W while the values at the vertices x_N and x_S are computed as the arithmetic mean of values on finite volumes which are common to this vertex (for general nonuniform meshes see [1]). After a short calculation we are ready to write

$$p_\sigma = \begin{pmatrix} \frac{u_E-u_W}{h} \\ \frac{u_N-u_S}{h} \end{pmatrix} = \frac{u_E-u_W}{h}\mathbf{n}_{K,\sigma} + \frac{u_N-u_S}{h}\mathbf{t}_{K,\sigma}, \tag{8}$$

where $\mathbf{t}_{K,\sigma}$ is a unit vector parallel to σ such that $(x_N - x_S)\cdot\mathbf{t}_{K,\sigma} > 0$ and and h is a length of an edge σ. Finally, applying (8) in (5), we get

$$\phi_\sigma = a_\sigma \frac{u_E-u_W}{h} + b_\sigma \frac{u_N-u_S}{h}. \tag{9}$$

where $D_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ b_\sigma & c_\sigma \end{pmatrix}$ is matrix D written in the basis $(\mathbf{n}_{K,\sigma}, \mathbf{t}_{K,\sigma})$. In such a way, the finite volume scheme is transformed into a linear system of equations.

In our presentation we show application of the scheme to real, e.g. medical images and discuss the noise removal and connecting of interupted structures.

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