Najděte aproximaci řešení úlohy $-0.4y''+y'=2$; $y(0)=-1$

$-0.4y'(1)=2y(1)$

s krokem $\frac{1}{4}$.

Řešení:

$l=1$, $h=\frac{1}{4} \Rightarrow n=\frac{l}{h}=\frac{1}{\frac{1}{4}}=4 \Rightarrow i=0,1,2,3,4$

$x_i=i\cdot h$, $i=0,\dots,4$

$x_i:\ 0;\ \frac{1}{4};\ \frac{1}{2};\ \frac{3}{4};\ 1$

- prostor přípustných variací: $V=\{v\in \mathcal{H}^1(0,1):\ v(0)=0\}$

  (má hodnotu 0 v těch krajních bodech, v nichž je hodnota hledané funkce dána)
- množina přípustných řešení: $W=\{w\in \mathcal{H}^1(0,1):\ w(0)=-1\}$

$$-0.4\,y''+y'=2 \qquad /\cdot v$$

$$-0.4\,y''v+y'v=2v \qquad /\int_0^1 \dots dx$$

$$\int_0^1(-0.4\,y''v+y'v)\,dx=\int_0^1 2v\,dx$$

$$\underbrace{\int_0^1(-0.4\,y''v)\,dx}_{\text{použijeme metodu per partes}}+\int_0^1 y'v\,dx=\int_0^1 2v\,dx \qquad (*)$$

$$\int_0^1(-0.4\,y''v)\,dx=\left|\begin{matrix} u=v & w'=-0.4\,y'' \\ u'=v' & w=-0.4\,y'\end{matrix}\right|=-0.4\,[v\cdot y']_0^1+0.4\int_0^1 v'y'\,dx=$$

$$=-0.4\,[v(1)\underbrace{y'(1)}_{\downarrow}-\underbrace{v(0)}_{=0}y'(0)]+0.4\int_0^1 v'y'\,dx=0.4\int_0^1 v'y'\,dx-0.4\cdot\frac{2}{-0.4}\,y(1)v(1)=$$

z podmínky $-0.4y'(1)=2y(1)$

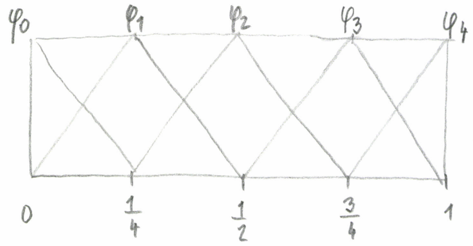
dostáváme: $y'(1)=\frac{2}{-0.4}\,y(1)$

$$=0.4\int_0^1 v'y'\,dx+2y(1)v(1)$$ ... dosadíme do $(*)$

$$0.4\int_0^1 y'v'\,dx+\int_0^1 y'v\,dx+2y(1)v(1)=\int_0^1 2v\,dx$$

$$\underbrace{\int_0^1(0.4\,y'v'+y'v)\,dx+2y(1)v(1)}_{B(y,v)}=\underbrace{\int_0^1 2v\,dx}_{L(v)}$$

Galerkinova formulace: Najděte $y\in W$ tak, aby bylo splněno $B(y,v)=L(v)$ $\forall v\in V$.



$$\varphi_0(x) = \begin{cases} -4x+1 & 0 \le x \le \frac{1}{4} \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_0'(x) = \begin{cases} -4 & 0 \le x \le \frac{1}{4} \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_1(x) = \begin{cases} 4x & 0 \le x \le \frac{1}{4} \\ -4x+2 & \frac{1}{4} \le x \le \frac{1}{2} \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_1'(x) = \begin{cases} 4 & 0 \le x \le \frac{1}{4} \\ -4 & \frac{1}{4} \le x \le \frac{1}{2} \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_2(x) = \begin{cases} 4x-1 & \frac{1}{4} \le x \le \frac{1}{2} \\ -4x+3 & \frac{1}{2} \le x \le \frac{3}{4} \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_2'(x) = \begin{cases} 4 & \frac{1}{4} \le x \le \frac{1}{2} \\ -4 & \frac{1}{2} \le x \le \frac{3}{4} \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_3(x) = \begin{cases} 4x-2 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4x+4 & \frac{3}{4} \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_3'(x) = \begin{cases} 4 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4 & \frac{3}{4} \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_4(x) = \begin{cases} 4x-3 & \frac{3}{4} \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_4'(x) = \begin{cases} 4 & \frac{3}{4} \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

$$B(\varphi_0,\varphi_0) = \int_0^{0.25}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+1)\right)dx = \int_0^{0.25}(6.4+16x-4)\,dx = \int_0^{0.25}(2.4-16x)\,dx =$$
(nepotřebujeme)

$$= 2.4\,[x]_0^{0.25} - \frac{16}{2}\,[x^2]_0^{0.25} = 2.4\cdot 0.25 - 8\cdot 0.25^2 = \underline{0.1}$$

$$B(\varphi_1,\varphi_1) = \int_0^{0.25}(0.4\cdot 4\cdot 4 + 4\cdot 4x)\,dx + \int_{0.25}^{0.5}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+2)\right)dx =$$

$$= \int_0^{0.25}(6.4+16x)\,dx + \int_{0.25}^{0.5}(6.4+16x-8)\,dx = 6.4\,[x]_0^{0.25} + \frac{16}{2}\,[x^2]_0^{0.25} - 1.6\,[x]_{0.25}^{0.5} + \frac{16}{2}\,[x^2]_{0.25}^{0.5} =$$

$$= 6.4\cdot 0.25 + 8\cdot 0.25^2 - 1.6\cdot(0.5-0.25) + 8(0.5^2-0.25^2) = \underline{3.2}$$

$$B(\varphi_2,\varphi_2) = \int_{0.25}^{0.5}\left(0.4\cdot 4\cdot 4 + 4\cdot(4x-1)\right)dx + \int_{0.5}^{0.75}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+3)\right)dx =$$

$$= \int_{0.25}^{0.5}(6.4+16x-4)\,dx + \int_{0.5}^{0.75}(6.4+16x-12)\,dx = 2.4\,[x]_{0.25}^{0.5} + \frac{16}{2}[x^2]_{0.25}^{0.5} - 5.6\,[x]_{0.5}^{0.75} + \frac{16}{2}[x^2]_{0.5}^{0.75} = \underline{3.2}$$

$$B(\varphi_3,\varphi_3) = \int_{0.5}^{0.75}\left(0.4\cdot 4\cdot 4 + 4\cdot(4x-2)\right)dx + \int_{0.75}^{1}\left(0.4\cdot(-4)\cdot(-4) + (-4)\cdot(-4x+4)\right)dx =$$

$$= \int_{0.5}^{0.75}(6.4+16x-8)\,dx + \int_{0.75}^{1}(+6.4+16x-16)\,dx = -1.6\,[x]_{0.5}^{0.75} + \frac{16}{2}[x^2]_{0.5}^{0.75} - 9.6\,[x]_{0.75}^{1} + \frac{16}{2}[x^2]_{0.75}^{1} = \underline{3.2}$$

$$B(\varphi_4,\varphi_4) = \int_{0.75}^{1}\left(0.4\cdot 4\cdot 4 + 4\cdot(4x-3)\right)dx + 2\,\underbrace{u(1)}_{\varphi_4(1)=1}\,\underbrace{v(1)}_{\varphi_4(1)=1} = \int_{0.75}^{1}(6.4+16x-12)\,dx + 2\cdot 1\cdot 1 =$$

$$= -5.6\,[x]_{0.75}^{1} + \frac{16}{2}[x^2]_{0.75}^{1} + 2 = \underline{4.1}$$

$$B(\varphi_0,\varphi_1) = \int_0^{0.25}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot 4x\right)dx = \int_0^{0.25}(-6.4-16x)\,dx = -6.4\,[x]_0^{0.25} - \frac{16}{2}[x^2]_0^{0.25} = \underline{-2.1}$$

$$B(\varphi_1,\varphi_2) = \int_{0.25}^{0.5}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot(4x-1)\right)dx = \int_{0.25}^{0.5}(-6.4-16x+4)\,dx = -2.4\,[x]_{0.25}^{0.5} - \frac{16}{2}[x^2]_{0.25}^{0.5} = \underline{-2.1}$$

$$B(\varphi_2,\varphi_3) = \int_{0.5}^{0.75}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot(4x-2)\right)dx = \int_{0.5}^{0.75}(-6.4-16x+8)\,dx = 1.6\,[x]_{0.5}^{0.75} - \frac{16}{2}[x^2]_{0.5}^{0.75} = \underline{-2.1}$$

$$B(\varphi_3,\varphi_4) = \int_{0.75}^{1}\left(0.4\cdot(-4)\cdot 4 + (-4)\cdot(4x-3)\right)dx = \int_{0.75}^{1}(-6.4-16x+12)\,dx = 5.6\,[x]_{0.75}^{1} - \frac{16}{2}[x^2]_{0.75}^{1} = \underline{-2.1}$$

$$B(\varphi_1,\varphi_0) = \int_0^{0.25}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+1)\right)dx = \int_0^{0.25}(-6.4-16x+4)\,dx = -2.4\,[x]_0^{0.25} - \frac{16}{2}[x^2]_0^{0.25} = \underline{-1.1}$$
(nepotřebujeme)

$$B(\varphi_2,\varphi_1) = \int_{0.25}^{0.5}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+2)\right)dx = \int_{0.25}^{0.5}(-6.4-16x+8)\,dx = 1.6\,[x]_{0.25}^{0.5} - \frac{16}{2}[x^2]_{0.25}^{0.5} = \underline{-1.1}$$

$$B(\varphi_3,\varphi_2) = \int_{0.5}^{0.75}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+3)\right)dx = \int_{0.5}^{0.75}(-6.4-16x+12)\,dx = 5.6\,[x]_{0.5}^{0.75} - \frac{16}{2}[x^2]_{0.5}^{0.75} = \underline{-1.1}$$

$$B(\varphi_4,\varphi_3) = \int_{0.75}^{1}\left(0.4\cdot 4\cdot(-4) + 4\cdot(-4x+4)\right)dx = \int_{0.75}^{1}(-6.4-16x+16)\,dx = 9.6\,[x]_{0.75}^{1} - \frac{16}{2}[x^2]_{0.75}^{1} = \underline{-1.1}$$

Řešíme $L(v) = \int_0^1 2v\,dx$

$$L(\varphi_0) = \int_0^{0.25} 2\cdot(-4x+1)\,dx = -\frac{8}{2}\left[x^2\right]_0^{0.25} + 2\left[x\right]_0^{0.25} = \underline{\underline{0.25}}$$

(nepotřebujeme)

$$L(\varphi_1) = \int_0^{0.25} 2\cdot(4x)\,dx + \int_{0.25}^{0.5} 2\cdot(-4x+2)\,dx = \frac{8}{2}\left[x^2\right]_0^{0.25} - \frac{8}{2}\left[x^2\right]_{0.25}^{0.5} + 4\left[x\right]_{0.25}^{0.5} = \underline{\underline{0.5}}$$

$$L(\varphi_2) = \int_{0.25}^{0.5} 2\cdot(4x-1)\,dx + \int_{0.5}^{0.75} 2\cdot(-4x+3)\,dx = \frac{8}{2}\left[x^2\right]_{0.25}^{0.5} - 2\left[x\right]_{0.25}^{0.5} - \frac{8}{2}\left[x^2\right]_{0.5}^{0.75} + 6\left[x\right]_{0.5}^{0.75} = \underline{\underline{0.5}}$$

$$L(\varphi_3) = \int_{0.5}^{0.75} 2(4x-2)\,dx + \int_{0.75}^{1} 2\cdot(-4x+4)\,dx = \frac{8}{2}\left[x^2\right]_{0.5}^{0.75} - 4\left[x\right]_{0.5}^{0.75} - \frac{8}{2}\left[x^2\right]_{0.75}^{1} + 8\left[x\right]_{0.75}^{1} = \underline{\underline{0.5}}$$

$$L(\varphi_4) = \int_{0.75}^{1} 2\cdot(4x-3)\,dx = \frac{8}{2}\left[x^2\right]_{0.75}^{1} - 6\left[x\right]_{0.75}^{1} = \underline{\underline{0.25}}$$

Maticový zápis soustavy lineárních rovnic:

$$\begin{pmatrix} B(\varphi_0,\varphi_0) & B(\varphi_1,\varphi_0) & 0 & 0 & 0 \\ B(\varphi_0,\varphi_1) & B(\varphi_1,\varphi_1) & B(\varphi_2,\varphi_1) & 0 & 0 \\ 0 & B(\varphi_1,\varphi_2) & B(\varphi_2,\varphi_2) & B(\varphi_3,\varphi_2) & 0 \\ 0 & 0 & B(\varphi_2,\varphi_3) & B(\varphi_3,\varphi_3) & B(\varphi_4,\varphi_3) \\ 0 & 0 & 0 & B(\varphi_3,\varphi_4) & B(\varphi_4,\varphi_4) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} L(\varphi_0) \\ L(\varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) \end{pmatrix}$$

Nyní uvažujeme okrajovou podmínku $y(0) = -1$ (ze zadání), tj. $y(0) = y_0 = -1 \Rightarrow$ první složku vektoru řešení $y = (y_0, y_1, \dots, y_4)^T$ známe $\Rightarrow$ 1. rovnici můžeme vynechat. Zbývá upravit 2. rovnici systému (rozepíši):

$$B(\varphi_0,\varphi_1)\cdot \underbrace{y_0}_{-1} + B(\varphi_1,\varphi_1)\cdot y_1 + B(\varphi_2,\varphi_1)\cdot y_2 = L(\varphi_1)$$

$\Rightarrow$ převedeme na pravou stranu a dostáváme:

$$B(\varphi_1,\varphi_1)\cdot y_1 + B(\varphi_2,\varphi_1)\cdot y_2 = L(\varphi_1) + B(\varphi_0,\varphi_1) = 0.5 - 2.1 = -1.6$$

Celkem řešíme systém:

$$\begin{pmatrix} B(\varphi_1,\varphi_1) & B(\varphi_2,\varphi_1) & 0 & 0 \\ B(\varphi_1,\varphi_2) & B(\varphi_2,\varphi_2) & B(\varphi_3,\varphi_2) & 0 \\ 0 & B(\varphi_2,\varphi_3) & B(\varphi_3,\varphi_3) & B(\varphi_4,\varphi_3) \\ 0 & 0 & B(\varphi_3,\varphi_4) & B(\varphi_4,\varphi_4) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \overbrace{L(\varphi_1) + B(\varphi_0,\varphi_1)}^{\text{díky okrajové podmínce}} \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) \end{pmatrix}$$

Po dosazení řešíme systém lin. rovnic:

$$\begin{pmatrix} 3.2 & -1.1 & 0 & 0 \\ -2.1 & 3.2 & -1.1 & 0 \\ 0 & -2.1 & 3.2 & -1.1 \\ 0 & 0 & -2.1 & 4.1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1.6 \\ 0.5 \\ 0.5 \\ 0.25 \end{pmatrix}$$

$\Rightarrow$ řešení

$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -0.5673 \\ -0.1956 \\ 0.0593 \\ 0.0913 \end{pmatrix}$$

$\rightarrow$ z okrajové (Dirichletovy) podmínky

přibližné řešení:

$$y_{B} = \sum_{i=0}^{n} y_i \varphi_i(x) = -\varphi_0(x) - 0.5673\,\varphi_1(x) - 0.1956\,\varphi_2(x) + 0.0593\,\varphi_3(x) + 0.0913\,\varphi_4(x)$$