

Uvažujme úlohu

$$-a^2 y'' + py' + qy = f \text{ pro } x \in (0, l) \quad (1)$$

a okrajové podmínky

- Dirichletovy:  $y(0) = \alpha_0$ ,  $y(l) = \alpha_l$ ,
- Neumannovy:  $a^2 y'(0) = \beta_0$ ,  $-a^2 y'(l) = \beta_l$ ,
- Newtonovy:  $a^2 y'(0) = \gamma_0 y(0) + \beta_0$ ,  $-a^2 y'(l) = \gamma_l y(l) + \beta_l$ ,

#### METODA KONEČNÝCH PRVKŮ

- prostor testovacích funkcí  $V$
- množina přípustných řešení  $W$
- Galerkinova (slabá) formulace úlohy: Najděte funkci  $y \in W$  tak, aby

$$B(y, v) = L(v) \quad \forall v \in V,$$

kde

$$B(y, v) = \int_0^l (-a^2 y'' + py' + qy) v \, dx \quad \text{a} \quad L(v) = \int_0^l f v \, dx$$

- diskretizace úlohy:  $n = \frac{l}{h}$ ,  $l$  - délka intervalu,  $h$  - diskretizační krok
- přibližné řešení:  $y_G = \sum_{j=0}^n c_j \varphi_j$ , neznámé koeficienty  $c_0, c_1, \dots, c_n$  dostáváme jako řešení soustavy lineárních rovnic

$$K \vec{c} = \vec{b}, \text{ kde}$$

$K$  se nazývá matice tuhosti a  $\vec{b}$  vektor zatížení.

$$\begin{pmatrix} B(\varphi_0, \varphi_0) & B(\varphi_1, \varphi_0) & \cdots & B(\varphi_n, \varphi_0) \\ B(\varphi_0, \varphi_1) & B(\varphi_1, \varphi_1) & \cdots & B(\varphi_n, \varphi_1) \\ \vdots & & \ddots & \vdots \\ B(\varphi_0, \varphi_n) & B(\varphi_1, \varphi_n) & \cdots & B(\varphi_n, \varphi_n) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} L(\varphi_0) \\ L(\varphi_1) \\ \vdots \\ L(\varphi_n) \end{pmatrix}$$

**Příklad 1.** Najděte aproximaci řešení úlohy

$$\begin{aligned} -0.4y'' + y' &= 2 \\ y(0) &= -1 \\ -0.4y'(1) &= 2y(1) \end{aligned}$$

s krokem  $h = \frac{1}{4}$ .

**Řešení.**

**Příklad 2.** Najděte aproximaci řešení úlohy

$$\begin{aligned} -y'' + 3y &= \sin x \\ y(0) &= 1 \\ y(1) &= 2 \end{aligned}$$

s krokem  $h = \frac{1}{5}$ .

**Řešení.**

$$l = 1, h = \frac{1}{5} \Rightarrow n = \frac{l}{h} = \frac{1}{0.2} = 5 \Rightarrow \begin{array}{c|cccccc} i & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline x_i & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \end{array}$$

- množina testovacích funkcí (prostor přípustných variací):  $V = \{v \in \mathcal{H}^1(0, 1) : v(0) = v(1) = 0\}$
- množina přípustných řešení:  $W = \{w \in \mathcal{H}^1(0, 1) : w(0) = 1, w(1) = 2\}$

$$\begin{aligned} -y'' + 3y &= \sin x \quad / \cdot v \\ -y''v + 3yv &= \sin x \cdot v \quad / \int_0^1 \dots dx \\ \int_0^1 (-y''v + 3yv) dx &= \int_0^1 \sin x \cdot v dx \\ \underbrace{\int_0^1 (-y''v) dx}_{\text{per partes}} + \int_0^1 3yv dx &= \int_0^1 \sin x \cdot v dx \end{aligned} \quad (2)$$

$$\begin{aligned} \int_0^1 (-y''v) dx &= \left| \begin{array}{cc} u = v & w' = -y'' \\ u' = v' & w = -y' \end{array} \right|_0^1 = [v \cdot (-y')]_0^1 + \int_0^1 v'y' dx = -\underbrace{v(1)}_{=0} \underbrace{y'(1)}_{=0} + \underbrace{v(0)}_{=0} \underbrace{y'(0)}_{=0} \\ &= \int_0^1 v'y' dx + \int_0^1 v'y' dx \quad \dots \text{dosadíme zpět do rovnice (2) a dostáváme} \end{aligned}$$

$$\underbrace{\int_0^1 v'y' dx + 3 \int_0^1 yv dx}_{B(y,v)} = \underbrace{\int_0^1 \sin x \cdot v dx}_{L(v)}$$

- Galerkinova (slabá) formulace úlohy:  
Najděte funkci  $y \in W$  tak, aby bylo splněno  $B(y, v) = L(v) \quad \forall v \in V$ , kde

$$B(y, v) = \int_0^1 v'y' dx + 3 \int_0^1 yv dx \quad \text{a} \quad L(v) = \int_0^1 \sin x \cdot v dx$$

- tvar slabého řešení:

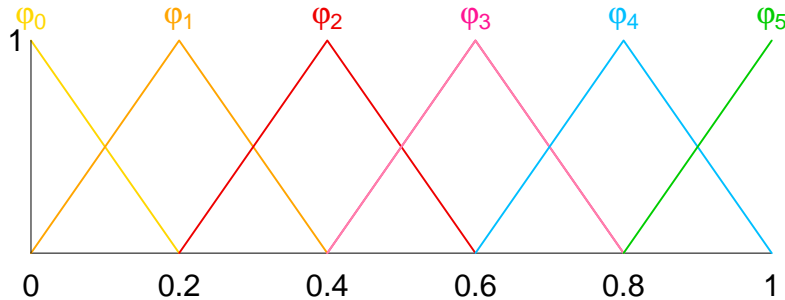
$$y_G(x) = \underbrace{y_0}_{=1} \varphi_0(x) + y_1 \varphi_1(x) + y_2 \varphi_2(x) + y_3 \varphi_3(x) + y_4 \varphi_4(x) + \underbrace{y_5}_{=2} \varphi_5(x) \in W$$

$$y_G(x) = \varphi_0(x) + y_1 \varphi_1(x) + y_2 \varphi_2(x) + y_3 \varphi_3(x) + y_4 \varphi_4(x) + 2\varphi_5(x) \Rightarrow \text{hledáme 4 neznámé } y_1, y_2, y_3, y_4$$

- maticový zápis soustavy lineárních rovnic:

$$\begin{pmatrix} B(\varphi_1, \varphi_1) & B(\varphi_2, \varphi_1) & \underbrace{B(\varphi_3, \varphi_1)}_{=0} & \underbrace{B(\varphi_4, \varphi_1)}_{=0} \\ B(\varphi_1, \varphi_2) & B(\varphi_2, \varphi_2) & B(\varphi_3, \varphi_2) & \underbrace{B(\varphi_4, \varphi_2)}_{=0} \\ \underbrace{B(\varphi_1, \varphi_3)}_{=0} & B(\varphi_2, \varphi_3) & B(\varphi_3, \varphi_3) & B(\varphi_4, \varphi_3) \\ \underbrace{B(\varphi_1, \varphi_4)}_{=0} & \underbrace{B(\varphi_2, \varphi_4)}_{=0} & B(\varphi_3, \varphi_4) & B(\varphi_4, \varphi_4) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} L(\varphi_1) - y_0 B(\varphi_0, \varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) - y_5 B(\varphi_5, \varphi_4) \end{pmatrix}$$

- testovací funkce



$$\begin{aligned} \varphi_0(x) &= \begin{cases} 1 - 5x & 0 \leq x \leq 0.2 \\ 0 & \text{jinak} \end{cases} & \varphi'_0(x) &= \begin{cases} -5 & 0 \leq x \leq 0.2 \\ 0 & \text{jinak} \end{cases} \\ \varphi_1(x) &= \begin{cases} 5x & 0 \leq x \leq 0.2 \\ 2 - 5x & 0.2 \leq x \leq 0.4 \\ 0 & \text{jinak} \end{cases} & \varphi'_1(x) &= \begin{cases} 5 & 0 \leq x \leq 0.2 \\ -5 & 0.2 \leq x \leq 0.4 \\ 0 & \text{jinak} \end{cases} \\ \varphi_2(x) &= \begin{cases} 5x - 1 & 0.2 \leq x \leq 0.4 \\ 3 - 5x & 0.4 \leq x \leq 0.6 \\ 0 & \text{jinak} \end{cases} & \varphi'_2(x) &= \begin{cases} 5 & 0.2 \leq x \leq 0.4 \\ -5 & 0.4 \leq x \leq 0.6 \\ 0 & \text{jinak} \end{cases} \\ \varphi_3(x) &= \begin{cases} 5x - 2 & 0.4 \leq x \leq 0.6 \\ 4 - 5x & 0.6 \leq x \leq 0.8 \\ 0 & \text{jinak} \end{cases} & \varphi'_3(x) &= \begin{cases} 5 & 0.4 \leq x \leq 0.6 \\ -5 & 0.6 \leq x \leq 0.8 \\ 0 & \text{jinak} \end{cases} \\ \varphi_4(x) &= \begin{cases} 5x - 3 & 0.6 \leq x \leq 0.8 \\ 5 - 5x & 0.8 \leq x \leq 1 \\ 0 & \text{jinak} \end{cases} & \varphi'_4(x) &= \begin{cases} 5 & 0.6 \leq x \leq 0.8 \\ -5 & 0.8 \leq x \leq 1 \\ 0 & \text{jinak} \end{cases} \\ \varphi_5(x) &= \begin{cases} 5x - 4 & 0.8 \leq x \leq 1 \\ 0 & \text{jinak} \end{cases} & \varphi'_5(x) &= \begin{cases} 5 & 0.8 \leq x \leq 1 \\ 0 & \text{jinak} \end{cases} \end{aligned}$$

- výpočet prvků matice tuhosti  $K$ :

$$\begin{aligned} B(\varphi_1, \varphi_1) &= \int_0^{0.2} (5 \cdot 5 + 3 \cdot 5x \cdot 5x) dx + \int_{0.2}^{0.4} (-5 \cdot (-5) + 3(2 - 5x)^2) dx = 10.4 \\ B(\varphi_2, \varphi_2) &= \int_{0.2}^{0.4} (5 \cdot 5 + 3(5x - 1)^2) dx + \int_{0.4}^{0.6} (-5 \cdot (-5) + 3(3 - 5x)^2) dx = 10.4 \\ B(\varphi_3, \varphi_3) &= \int_{0.4}^{0.6} (5 \cdot 5 + 3(5x - 2)^2) dx + \int_{0.6}^{0.8} (-5 \cdot (-5) + 3(4 - 5x)^2) dx = 10.4 \\ B(\varphi_4, \varphi_4) &= \int_{0.6}^{0.8} (5 \cdot 5 + 3(5x - 3)^2) dx + \int_{0.8}^1 (-5 \cdot (-5) + 3(5 - 5x)^2) dx = 10.4 \end{aligned}$$

$$B(\varphi_2, \varphi_1) = \int_{0.2}^{0.4} (5 \cdot (-5) + 3(5x - 1)(2 - 5x)) dx = -4.9$$

$$B(\varphi_3, \varphi_2) = \int_{0.4}^{0.6} (5 \cdot (-5) + 3(5x - 2)(3 - 5x)) dx = -4.9$$

$$B(\varphi_4, \varphi_3) = \int_{0.6}^{0.8} (5 \cdot (-5) + 3(5x - 3)(4 - 5x)) dx = -4.9$$

$$B(\varphi_1, \varphi_2) = \int_{0.2}^{0.4} (-5 \cdot 5 + 3(2 - 5x)(5x - 1)) dx = -4.9$$

$$B(\varphi_2, \varphi_3) = \int_{0.4}^{0.6} (-5 \cdot 5 + 3(3 - 5x)(5x - 2)) dx = -4.9$$

$$B(\varphi_3, \varphi_4) = \int_{0.6}^{0.8} (-5 \cdot 5 + 3(4 - 5x)(5x - 3)) dx = -4.9$$

- výpočet  $\tilde{L}(\varphi_1)$ :

$$\int_0^{0.2} \sin x \cdot 5x dx = \left| \begin{array}{l} u = 5x \quad v' = \sin x \\ u' = 5 \quad v = -\cos x \end{array} \right| = -5[x \cos x]_0^{0.2} + 5 \int_0^{0.2} \cos x dx = -5(0.2 \cos 0.2 - 0) + 5[\sin x]_0^{0.2}$$

$$= -\cos 0.2 + 5 \sin 0.2 = 0.0133$$

$$\int_{0.2}^{0.4} \sin x \cdot (2 - 5x) dx = 2[-\cos x]_{0.2}^{0.4} - 5 \int_{0.2}^{0.4} x \sin x dx = -2(\cos 0.4 - \cos 0.2) - 5 \left( [-x \cos x]_{0.2}^{0.4} + \int_{0.2}^{0.4} \cos x dx \right)$$

$$= -2 \cos 0.4 + 2 \cos 0.2 - 5(-0.4 \cos 0.4 + 0.2 \cos 0.2) - 5[\sin x]_{0.2}^{0.4} = 0.0263$$

$$L(\varphi_1) = \int_0^{0.2} \sin x \cdot 5x dx + \int_{0.2}^{0.4} \sin x \cdot (2 - 5x) dx = 0.0133 + 0.0263 = 0.0396$$

$$B(\varphi_0, \varphi_1) = \int_0^{0.2} (-5 \cdot 5 + 3(1 - 5x) \cdot 5x) dx = -4.9$$

$$\tilde{L}(\varphi_1) = L(\varphi_1) - y_0 B(\varphi_0, \varphi_1) = 0.0396 - 1 \cdot (-4.9) = 4.9396$$

- výpočet  $\tilde{L}(\varphi_4)$ :

$$L(\varphi_4) = \int_{0.6}^{0.8} \sin x (5x - 3) dx + \int_{0.8}^1 \sin x \cdot (5 - 5x) dx = 10 \sin 0.8 - 5 \sin 0.6 + 5 \sin 1 = 0.1430$$

$$B(\varphi_5, \varphi_4) = \int_{0.8}^1 (5 \cdot (-5) + 3(5x - 4)(5 - 5x)) dx = -4.9$$

$$\hat{L}(\varphi_4) = L(\varphi_4) - y_5 B(\varphi_5, \varphi_4) = 0.1430 - 2 \cdot (-4.9) = 9.943$$

- výpočet  $L(\varphi_2)$ ,  $L(\varphi_3)$ :

$$L(\varphi_2) = \int_{0.2}^{0.4} \sin x \cdot (5x - 1) \, dx + \int_{0.4}^{0.6} \sin x \cdot (3 - 5x) \, dx = 10 \sin 0.4 - 5 \sin 0.2 + 5 \sin 0.6 = 0.0776$$

$$L(\varphi_3) = \int_{0.4}^{0.6} \sin x \cdot (5x - 2) \, dx + \int_{0.4}^{0.6} \sin x \cdot (4 - 5x) \, dx = 10 \sin 0.6 - 5 \sin 0.4 + 5 \sin 0.8 = 0.1126$$

- soustava lineárních rovnic:

$$\begin{pmatrix} 10.4 & -4.9 & 0 & 0 \\ -4.9 & 10.4 & -4.9 & 0 \\ 0 & -4.9 & 10.4 & -4.9 \\ 0 & 0 & -4.9 & 10.4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 4.9396 \\ 0.0776 \\ 0.1126 \\ 9.943 \end{pmatrix}$$

- řešení soustavy lineárních rovnic:

$$\vec{y} = (1, 0.9632, 1.0363, 1.2205, 1.5311, 2)^T$$

- přibližné řešení diferenciální rovnice:

$$y_G = \varphi_0 + 0.9632\varphi_1 + 1.0363\varphi_2 + 1.2205\varphi_3 + 1.5311\varphi_4 + 2\varphi_5$$