Uvažujme úlohu

$$-a^2y'' + py' + qy = f \text{ pro } x \in (0, l) \tag{1}$$

a okrajové podmínky

- Dirichletovy: $y(0) = \alpha_0$, $y(l) = \alpha_l$,
- Neumannovy: $a^2y'(0) = \beta_0$, $-a^2y'(l) = \beta_l$,
- Newtonovy: $a^2y'(0) = \gamma_0 y(0) + \beta_0$, $-a^2y'(l) = \gamma_l y(l) + \beta_l$,

METODA KONEČNÝCH PRVKŮ

- prostor testovacích funkcí V
- množina přípustných řešení W
- Galerkinova (slabá) formulace úlohy: Najděte funkci $y \in W$ tak, aby

  $$B(y, v) = L(v) \quad \forall v \in V,$$

  kde

  $$B(y, v) = \int_0^l \left(-a^2y'' + py' + qy\right) v \, \mathrm{d}x \qquad \text{a} \qquad L(v) = \int_0^l f v \, \mathrm{d}x$$

- diskretizece úlohy: $n = \frac{l}{h}$, l - délka intervalu, h - diskretizační krok
- přibližné řešení: $y_G = \sum\limits_{j=0}^{n} c_j \varphi_j$, neznámé koeficienty $c_0, c_1, \ldots, c_n$ dostáváme jako řešení soustavy lineárních rovnic

  $$K\overrightarrow{c} = \overrightarrow{b}, \text{ kde}$$

  K se nazývá matice tuhosti a $\overrightarrow{b}$ vektor zatížení.

  $$\begin{pmatrix} B(\varphi_0, \varphi_0) & B(\varphi_0, \varphi_1) & \cdots & B(\varphi_0, \varphi_n) \\ B(\varphi_1, \varphi_0) & B(\varphi_1, \varphi_1) & \cdots & B(\varphi_1, \varphi_n) \\ \vdots & & \ddots & \vdots \\ B(\varphi_n, \varphi_0) & B(\varphi_n, \varphi_1) & \cdots & B(\varphi_n, \varphi_n) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} L(\varphi_0) \\ L(\varphi_1) \\ \vdots \\ L(\varphi_n) \end{pmatrix}$$

Příklad 1. Najděte aproximaci řešení úlohy

$$\begin{aligned} -0.4y'' + y' &= 2 \\ y(0) &= -1 \\ -0.4y'(1) &= 2y(1) \end{aligned}$$

s krokem $h = \frac{1}{4}$.

Řešení.

Příklad 2. Najděte aproximaci řešení úlohy

$$\begin{aligned} -y'' + 3y &= \sin x \\ y(0) &= 1 \\ y(1) &= 2 \end{aligned}$$

s krokem $h = \frac{1}{5}$.

Řešení.

$$l = 1, h = \frac{1}{5} \Rightarrow n = \frac{l}{h} = \frac{1}{0.2} = 5 \Rightarrow x_i, i = 0, 1, \dots, 5 \Rightarrow x_i : 0, 0.2, 0.4, 0.6, 0.8, 1$$

- množina testovacích funkcí (prostor přípustných variací): $V = \left\{ v \in \mathcal{H}^1(0,1) : v(0) = v(1) = 0 \right\}$
- množina přípustných řešení: $W = \left\{ w \in \mathcal{H}^1(0,1) : w(0) = 1, w(1) = 2 \right\}$

$$-y'' + 3y = \sin x \quad / \cdot v$$

$$-y''v + 3yv = \sin x \cdot v \quad / \int_0^1 \dots \,\mathrm{d}x$$

$$\int_0^1 (-y''v + 3yv)\,\mathrm{d}x = \int_0^1 \sin x \cdot v \,\mathrm{d}x$$

$$\underbrace{\int_0^1 (-y''v)\,\mathrm{d}x}_{\text{per partes}} + \int_0^1 3yv\,\mathrm{d}x = \int_0^1 \sin x \cdot v \,\mathrm{d}x \tag{2}$$

$$\int_0^1 (-y''v)\,\mathrm{d}x = \begin{vmatrix} u = v & w' = -y'' \\ u' = v' & w = -y' \end{vmatrix} = [v \cdot (-y')]_0^1 + \int_0^1 v'y'\,\mathrm{d}x = -\underbrace{v(1)}_{=0} y'(1) + \underbrace{v(0)}_{=0} y'(0)$$

$$= \int_0^1 v'y'\,\mathrm{d}x + \int_0^1 v'y'\,\mathrm{d}x \qquad \text{\dots dosadíme zpět do rovnice (2) a dostáváme}$$

$$\underbrace{\int_0^1 v'y'\,\mathrm{d}x + 3\int_0^1 yv\,\mathrm{d}x}_{B(y,v)} = \underbrace{\int_0^1 \sin x \cdot v\,\mathrm{d}x}_{L(v)}$$

- Galerkinova (slabá) formulace úlohy:
  Najděte funkci $y \in W$ tak, aby bylo splněno $B(y,v) = L(v) \quad \forall v \in V$, kde

$$B(y,v) = \int_0^1 v'y'\,\mathrm{d}x + 3\int_0^1 yv\,\mathrm{d}x \qquad \text{a} \qquad L(v) = \int_0^1 \sin x \cdot v\,\mathrm{d}x$$

- tvar slabého řešení:

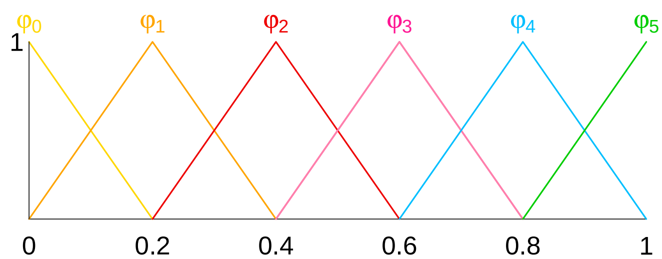
$$y_G(x) = \underbrace{y_0}_{=1} \varphi_0(x) + y_1\varphi_1(x) + y_2\varphi_2(x) + y_3\varphi_3(x) + y_4\varphi_4(x) + \underbrace{y_5}_{=2} \varphi_5(x) \in W$$

$$y_G(x) = \varphi_0(x) + y_1\varphi_1(x) + y_2\varphi_2(x) + y_3\varphi_3(x) + y_4\varphi_4(x) + 2\varphi_5(x) \quad \Rightarrow \text{ hledáme 4 neznámé } y_1, y_2, y_3, y_4$$

- maticový zápis soustavy lineárních rovnic:

$$\begin{pmatrix} B(\varphi_1,\varphi_1) & B(\varphi_2,\varphi_1) & \underbrace{B(\varphi_3,\varphi_1)}_{=0} & \underbrace{B(\varphi_4,\varphi_1)}_{=0} \\ B(\varphi_1,\varphi_2) & B(\varphi_2,\varphi_2) & B(\varphi_3,\varphi_2) & \underbrace{B(\varphi_4,\varphi_2)}_{=0} \\ \underbrace{B(\varphi_1,\varphi_3)}_{=0} & B(\varphi_2,\varphi_3) & B(\varphi_3,\varphi_3) & B(\varphi_4,\varphi_3) \\ \underbrace{B(\varphi_1,\varphi_4)}_{=0} & \underbrace{B(\varphi_2,\varphi_4)}_{=0} & B(\varphi_3,\varphi_4) & B(\varphi_4,\varphi_4) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} L(\varphi_1) - y_0 B(\varphi_0,\varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) \\ L(\varphi_4) - y_5 B(\varphi_5,\varphi_4) \end{pmatrix}$$

- testovací funkce



$$\varphi_0(x) = \begin{cases} 1-5x & 0 \le x \le 0.2 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_0'(x) = \begin{cases} -5 & 0 \le x \le 0.2 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_1(x) = \begin{cases} 5x & 0 \le x \le 0.2 \\ 2-5x & 0.2 \le x \le 0.4 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_1'(x) = \begin{cases} 5 & 0 \le x \le 0.2 \\ -5 & 0.2 \le x \le 0.4 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_2(x) = \begin{cases} 5x-1 & 0.2 \le x \le 0.4 \\ 3-5x & 0.4 \le x \le 0.6 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_2'(x) = \begin{cases} 5 & 0.2 \le x \le 0.4 \\ -5 & 0.4 \le x \le 0.6 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_3(x) = \begin{cases} 5x-2 & 0.4 \le x \le 0.6 \\ 4-5x & 0.6 \le x \le 0.8 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_3'(x) = \begin{cases} 5 & 0.4 \le x \le 0.6 \\ -5 & 0.6 \le x \le 0.8 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_4(x) = \begin{cases} 5x-3 & 0.6 \le x \le 0.8 \\ 5-5x & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_4'(x) = \begin{cases} 5 & 0.6 \le x \le 0.8 \\ -5 & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

$$\varphi_5(x) = \begin{cases} 5x-4 & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases} \qquad \varphi_5'(x) = \begin{cases} 5 & 0.8 \le x \le 1 \\ 0 & \text{jinak} \end{cases}$$

- výpočet prvků matice tuhosti K:

$$B(\varphi_1,\varphi_1) = \int_0^{0.2} (5 \cdot 5 + 3 \cdot 5x \cdot 5x)\,\mathrm{d}x + \int_{0.2}^{0.4} \left(-5 \cdot (-5) + 3\,(2-5x)^2\right)\,\mathrm{d}x = 10.4$$

$$B(\varphi_2,\varphi_2) = \int_{0.2}^{0.4} \left(5 \cdot 5 + 3\,(5x-1)^2\right)\,\mathrm{d}x + \int_{0.4}^{0.6} \left(-5 \cdot (-5) + 3\,(3-5x)^2\right)\,\mathrm{d}x = 10.4$$

$$B(\varphi_3,\varphi_3) = \int_{0.4}^{0.6} \left(5 \cdot 5 + 3\,(5x-2)^2\right)\,\mathrm{d}x + \int_{0.6}^{0.8} \left(-5 \cdot (-5) + 3\,(4-5x)^2\right)\,\mathrm{d}x = 10.4$$

$$B(\varphi_4,\varphi_4) = \int_{0.6}^{0.8} \left(5 \cdot 5 + 3\,(5x-3)^2\right)\,\mathrm{d}x + \int_{0.8}^{1} \left(-5 \cdot (-5) + 3\,(5-5x)^2\right)\,\mathrm{d}x = 10.4$$

$$B(\varphi_2,\varphi_1)=\int\limits_{0.2}^{0.4}\left(5\cdot(-5)+3\,(5x-1)\,(2-5x)\right)\,\mathrm{d}x=-4.9$$
$$B(\varphi_3,\varphi_2)=\int\limits_{0.4}^{0.6}\left(5\cdot(-5)+3\,(5x-2)\,(3-5x)\right)\,\mathrm{d}x=-4.9$$
$$B(\varphi_4,\varphi_3)=\int\limits_{0.6}^{0.8}\left(5\cdot(-5)+3\,(5x-3)\,(4-5x)\right)\,\mathrm{d}x=-4.9$$
$$B(\varphi_1,\varphi_2)=\int\limits_{0.2}^{0.4}\left(-5\cdot5+3\,(2-5x)\,(5x-1)\right)\,\mathrm{d}x=-4.9$$
$$B(\varphi_2,\varphi_3)=\int\limits_{0.4}^{0.6}\left(-5\cdot5+3\,(3-5x)\,(5x-2)\right)\,\mathrm{d}x=-4.9$$
$$B(\varphi_3,\varphi_4)=\int\limits_{0.6}^{0.8}\left(-5\cdot5+3\,(4-5x)\,(5x-3)\right)\,\mathrm{d}x=-4.9$$

- výpočet $\widetilde{L}(\varphi_1)$:

$$\int\limits_{0}^{0.2}\sin x\cdot 5x\,\mathrm{d}x=\left|\begin{array}{cc} u=5x & v'=\sin x\\ u'=5 & v=-\cos x\end{array}\right|=-5\left[x\cos x\right]_0^{0.2}+5\int\limits_{0}^{0.2}\cos x\,\mathrm{d}x=-5\,(0.2\cos 0.2-0)+5\left[\sin x\right]_0^{0.2}$$
$$=-\cos 0.2+5\sin 0.2=0.0133$$
$$\int\limits_{0.2}^{0.4}\sin x\cdot(2-5x)\,\mathrm{d}x=2\left[-\cos x\right]_{0.2}^{0.4}-5\int\limits_{0.2}^{0.4}x\sin x\,\mathrm{d}x=-2\,(\cos 0.4-\cos 0.2)-5\left(\left[-x\cos x\right]_{0.2}^{0.4}+\int\limits_{0.2}^{0.4}\cos x\,\mathrm{d}x\right)$$
$$=-2\cos 0.4+2\cos 0.2-5\,(-0.4\cos 0.4+0.2\cos 0.2)-5\left[\sin x\right]_{0.2}^{0.4}=0.0263$$
$$L(\varphi_1)=\int\limits_{0}^{0.2}\sin x\cdot 5x\,\mathrm{d}x+\int\limits_{0.2}^{0.4}\sin x\cdot(2-5x)\,\mathrm{d}x=0.0133+0.0263=0.0396$$
$$B(\varphi_0,\varphi_1)=\int\limits_{0}^{0.2}\left(-5\cdot5+3\,(1-5x)\cdot 5x\right)\,\mathrm{d}x=-4.9$$
$$\widetilde{L}(\varphi_1)=L(\varphi_1)-y_0B(\varphi_0,\varphi_1)=0.0396-1\cdot(-4.9)=4.9396$$

- výpočet $\widetilde{L}(\varphi_4)$:

$$L(\varphi_4)=\int\limits_{0.6}^{0.8}\sin x\,(5x-3)\,\mathrm{d}x+\int\limits_{0.8}^{1}\sin x\cdot(5-5x)\,\mathrm{d}x=10\sin 0.8-5\sin 0.6+5\sin 1=0.1430$$
$$B(\varphi_5,\varphi_4)=\int\limits_{0.8}^{1}\left(5\cdot(-5)+3\,(5x-4)\,(5-5x)\right)\,\mathrm{d}x=-4.9$$
$$\widehat{L}(\varphi_4)=L(\varphi_4)-y_5B(\varphi_5,\varphi_4)=0.1430-2\cdot(-4.9)=9.943$$

- výpočet $L(\varphi_2)$, $L(\varphi_3)$:

$$L(\varphi_2) = \int_{0.2}^{0.4} \sin x \cdot (5x-1)\ \mathrm{d}x + \int_{0.4}^{0.6} \sin x \cdot (3-5x)\ \mathrm{d}x = 10\sin 0.4 - 5\sin 0.2 + 5\sin 0.6 = 0.0776$$

$$L(\varphi_3) = \int_{0.4}^{0.6} \sin x \cdot (5x-2)\ \mathrm{d}x + \int_{0.4}^{0.6} \sin x \cdot (4-5x)\ \mathrm{d}x = 10\sin 0.6 - 5\sin 0.4 + 5\sin 0.8 = 0.1126$$

- soustava lineárních rovnic:

$$\begin{pmatrix} 10.4 & -4.9 & 0 & 0 \\ -4.9 & 10.4 & -4.9 & 0 \\ 0 & -4.9 & 10.4 & -4.9 \\ 0 & 0 & -4.9 & 10.4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 4.9396 \\ 0.0776 \\ 0.1126 \\ 9.943 \end{pmatrix}$$

- řešení soustavy lineárních rovnic:

$$\overrightarrow{y} = (1, 0.9632, 1.0363, 1.2205, 1.5311, 2)^T$$

- přibližné řešení diferenciální rovnice:

$$y_G = \varphi_0 + 0.9632\varphi_1 + 1.0363\varphi_2 + 1.2205\varphi_3 + 1.5311\varphi_4 + 2\varphi_5$$