Řešte integrály:

1. $\displaystyle\int \frac{x^3+x-1}{x^2+1}\,\mathrm{d}x$

Řešení.
Řešíme úpravou nebo dělením polynomů:

$$\int \frac{x^3+x-1}{x^2+1}\,\mathrm{d}x = \int \frac{x\left(x^2+1\right)-1}{x^2+1}\,\mathrm{d}x = \int \left(x - \frac{1}{x^2+1}\right)\mathrm{d}x = \frac{1}{2}x^2 - \mathrm{arctg}x + c$$

2. $\displaystyle\int \frac{x^4-1}{x+1}\,\mathrm{d}x$

Řešení.
Řešíme rozkladem podle vzorce a^2-b^2 nebo dělením polynomů:

$$\int \frac{x^4-1}{x+1}\,\mathrm{d}x = \int \frac{\left(x^2-1\right)\left(x^2+1\right)}{x+1}\,\mathrm{d}x = \int \frac{(x-1)(x+1)\left(x^2+1\right)}{x+1}\,\mathrm{d}x = \int (x-1)\left(x^2+1\right)\mathrm{d}x$$
$$= \int \left(x^3-x^2+x-1\right)\mathrm{d}x = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - x + c$$

3. $\displaystyle\int \frac{3x-1}{x^2+x+1}\,\mathrm{d}x$

Řešení.

$$\int \frac{3x-1}{x^2+x+1}\,\mathrm{d}x = \int \frac{\frac{3}{2}(2x+1)-\frac{5}{2}}{x^2+x+1}\,\mathrm{d}x = \int \left(\frac{3}{2}\cdot\frac{2x+1}{x^2+x+1} - \frac{5}{2}\frac{1}{x^2+x+1}\right)\mathrm{d}x$$

Úprava 2. integrálu - integrál typu $\dfrac{A}{ax^2+bx+c}$, kde $D=b^2-4ac<0 \Rightarrow$ integrál vedoucí na arctg

- úprava na čtverec: $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$
- úprava na arctg: $\left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \frac{3}{4}\left[1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2\right] = \frac{3}{4}\left[1+\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2\right]$

$$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x - \frac{5}{2}\int \frac{1}{\frac{3}{4}\left[1+\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2\right]}\,\mathrm{d}x = \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x - \frac{5}{2}\cdot\frac{4}{3}\int \frac{1}{1+\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2}\,\mathrm{d}x$$
$$= \frac{3}{2}\ln\left|x^2+x+1\right| - \frac{10}{3}\cdot\frac{\sqrt{3}}{2}\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c = \frac{3}{2}\ln\left|x^2+x+1\right| - \frac{5\sqrt{3}}{3}\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

4. $\displaystyle\int \frac{4x+3}{x^2-2x+3}\,\mathrm{d}x$

Řešení.

$$\int \frac{4x+3}{x^2-2x+3}\,\mathrm{d}x = \int \frac{2(2x-2)+7}{x^2-2x+3}\,\mathrm{d}x = \int \left(2\frac{2x-2}{x^2-2x+3} + \frac{7}{x^2-2x+3}\right)\mathrm{d}x$$

Úprava 2. integrálu - integrál typu $\dfrac{A}{ax^2+bx+c}$, kde $D=b^2-4ac<0 \Rightarrow$ integrál vedoucí na arctg

- úprava na čtverec: $x^2-2x+3 = (x-1)^2+2$
- úprava na arctg: $(x-1)^2+2 = 2\left[1+\frac{1}{2}(x-1)^2\right] = 2\left[1+\left(\frac{x-1}{\sqrt{2}}\right)^2\right]$

$$= \int \left(2\frac{2x-2}{x^2-2x+3} + \frac{7}{2\left[1+\left(\frac{x-1}{\sqrt{2}}\right)^2\right]} \right) \mathrm{d}x = \int \left(2\frac{2x-2}{x^2-2x+3} + \frac{7}{2} \cdot \frac{1}{1+\left(\frac{x-1}{\sqrt{2}}\right)^2} \right) \mathrm{d}x$$

$$= 2\ln\left|x^2-2x+3\right| + \frac{7\sqrt{2}}{2}\operatorname{arctg}\left(\frac{x-1}{\sqrt{2}}\right) + c$$

5. $\displaystyle\int \frac{4x+3}{\sqrt{x^2+2x+1}}\,\mathrm{d}x$

Řešení.

$$\int \frac{4x+3}{\sqrt{x^2+2x+1}}\,\mathrm{d}x = \int \frac{4x+3}{\sqrt{(x+1)^2}}\,\mathrm{d}x = \int \frac{4x+3}{x+1}\,\mathrm{d}x = |\text{dělení polynomů nebo úprava}|$$

$$= \int \frac{4(x+1)-1}{x+1}\,\mathrm{d}x = \int \left(4 - \frac{1}{x+1}\right) \mathrm{d}x = 4x - \ln|x+1| + c$$

6. $\displaystyle\int \frac{4x+3}{\sqrt{x^2+x-6}}\,\mathrm{d}x$

Řešení.

$$\int \frac{4x+3}{\sqrt{x^2+x-6}}\,\mathrm{d}x = \int \frac{2(2x+1)+1}{\sqrt{x^2+x-6}}\,\mathrm{d}x = 2\underbrace{\int \frac{2x+1}{\sqrt{x^2+x-6}}\,\mathrm{d}x}_{I_1} + \underbrace{\frac{1}{\sqrt{x^2+x-6}}\,\mathrm{d}x}_{I_2}$$

$$I_1 = \int \frac{2x+1}{\sqrt{x^2+x-6}}\,\mathrm{d}x = \left|\text{integrál typu } \int \frac{f'}{\sqrt{f}}\,\mathrm{d}x, \text{ substituce: } x^2+x-6=t^2\right| = 2\int \mathrm{d}\,t = 2t + c$$

$$= 2\sqrt{x^2+x-6} + c$$

$$I_2 = \int \frac{1}{\sqrt{x^2+x-6}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x+3)(x-2)}}\,\mathrm{d}x = \int \frac{1}{(x-2)\sqrt{\frac{x+3}{x-2}}}\,\mathrm{d}x = \left|\frac{x+3}{x-2} = t^2;\ \mathrm{d}x = -\frac{2}{5}t(x-2)^2\,\mathrm{d}t; x = \frac{2t^2+3}{t^2-1}\right|$$

$$= \int \frac{1}{t} \cdot \frac{-2}{5}t\left(\frac{2t^2+3}{t^2-1} - 2\right)\mathrm{d}t = -2\int \frac{1}{t^2-1}\,\mathrm{d}t = |\text{rozklad na parciální zlomky}| = -2\int \left(\frac{1}{2(t-1)} - \frac{1}{2(t+1)}\right)\mathrm{d}t$$

$$= -\ln|t-1| + \ln|t+1| + c = \ln\left|\frac{t+1}{t-1}\right| + c, \text{ kde } t = \sqrt{\frac{x+3}{x-2}}$$

Celkem dostáváme:

$$\int \frac{4x+3}{\sqrt{x^2+x-6}}\,\mathrm{d}x = 2I_1 + I_2 = 4\sqrt{x^2+x-6} + \ln\left|\frac{t+1}{t-1}\right| + c, \text{ kde } t = \sqrt{\frac{x+3}{x-2}}$$

7. $\displaystyle\int \frac{4x+3}{\sqrt{x^2-2x+3}}\,\mathrm{d}x$

Řešení.

$$\int \frac{4x+3}{\sqrt{x^2-2x+3}}\,\mathrm{d}x = \int \frac{2(2x-2)+7}{\sqrt{x^2-2x+3}}\,\mathrm{d}x = 2\underbrace{\int \frac{2x-2}{\sqrt{(x^2-2x+3)}}\,\mathrm{d}x}_{I_1} + 7\underbrace{\int \frac{1}{\sqrt{x^2-2x+3}}\,\mathrm{d}x}_{I_2}$$

$$I_1 = \int \frac{2x-2}{\sqrt{x^2-2x+3}}\,\mathrm{d}x = \left| x^2-2x+3 = t^2;\ \mathrm{d}x = \frac{2t}{2x-2}\,\mathrm{d}t \right| = \int \frac{2t}{t}\,\mathrm{d}t = 2t + c = 2\sqrt{x^2-2x+3} + c$$

$$I_2 = \int \frac{1}{\sqrt{x^2-2x+3}}\,\mathrm{d}x = \Bigg| D = -8 < 0 \Rightarrow \text{úprava na integrál typu } \int \frac{1}{\sqrt{x^2+1}}dx$$

$$\text{úprava na čtverec: } x^2-2x+3 = (x-1)^2+2 = 2\left(\left(\frac{x-1}{\sqrt{2}}\right)^2+1\right)\Bigg| = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}}\,\mathrm{d}x$$

$$= \left| \text{aplikace vztahu } \int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2+1}\right| + c \text{ (lze odvodit pomocí Eulerovy substituce - viz cvičení)} \right|$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \ln\left| \frac{x-1}{\sqrt{2}} + \sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1} \right| + c$$

Celkem dostáváme:

$$\int \frac{4x+3}{\sqrt{x^2-2x+3}}\,\mathrm{d}x = 2I_1 + 7I_2 = 4\sqrt{x^2-2x+3} + 7\ln\left| \frac{x-1}{\sqrt{2}} + \sqrt{\left(\frac{x-1}{\sqrt{2}}\right)^2+1} \right| + c$$

8. $\int \sqrt{x^2+1}\,\mathrm{d}x$

Řešení.
Zavedeme substituci: $\sqrt{x^2+1}+x = t$; $\mathrm{d}x = \frac{\sqrt{x^2+1}}{t}\,\mathrm{d}t$; $x = \frac{t^2-1}{2t}$

$$\int \sqrt{x^2+1}\,\mathrm{d}x = \int \frac{1}{t}\left(\frac{(t^2-1)^2}{4t^2}+1\right)\mathrm{d}t = \int \frac{1}{t}\cdot\frac{t^4+2t^2+1}{4t^2}\,\mathrm{d}t = \frac{1}{4}\int \frac{t^4+2t^2+1}{t^3}\,\mathrm{d}t = \frac{1}{4}\int\left(t+\frac{2}{t}+\frac{1}{t^3}\right)\mathrm{d}t$$

$$= \frac{1}{4}\left(\frac{1}{2}t^2 - \frac{1}{2t^2} + 2\ln|t|\right) + c = \frac{1}{8}\left(\left(\sqrt{x^2+1}+x\right)^2 - \frac{1}{\left(\sqrt{x^2+1}+x\right)^2}\right) + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right| + c$$

9. $\int \sqrt{1-x^2}\,\mathrm{d}x$

Řešení.
Zavedeme substituci: $x = \sin t$; $\mathrm{d}x = \cos t\,\mathrm{d}t$

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \int \sqrt{1-\sin^2 t}\cos t\,\mathrm{d}t = \int \sqrt{\cos^2 t}\cos t\,\mathrm{d}t = \int \cos^2 t\,\mathrm{d}t = \int \frac{1}{2}\left(1+\cos(2t)\right)\mathrm{d}t$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right) + c = |\text{využijeme vztah } \sin(2t) = 2\sin t\cos t| = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + c$$

Poznámka. Lze použít i substituci $x = \cos t$.

Soubor může obsahovat chyby. Pokud nějaké najdete, neváhejte mi je nahlásit :)