KŘIVKOVÝ INTEGRÁL DRUHÉHO DRUHU
(VE VEKTOROVÉM POLI)

- přes rovinnou křivku:

$$\int_{\varphi} P(x,y)\,dx + Q(x,y)\,dy = \int_a^b [P(\varphi(t),\psi(t))\cdot\varphi'(t) + Q(\varphi(t),\psi(t))\cdot\psi'(t)]\,dt$$

$$\varphi: \begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \\ t &\in \langle a,b\rangle \end{aligned}$$

- přes prostorovou křivku:

$$\int_{\varphi} P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz =$$

$$= \int_a^b [P(\varphi(t),\psi(t),\rho(t))\cdot\varphi'(t) + Q(\varphi(t),\psi(t),\rho(t))\cdot\psi'(t) + R(\varphi(t),\psi(t),\rho(t))\cdot\rho'(t)]\,dt$$

$$\varphi: \begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \\ z &= \rho(t) \\ t &\in \langle a,b\rangle \end{aligned}$$

- křivka φ je vždy orientovaná.
- Pokud křivka φ není souhlasně orientovaná s parametrizací, musíme před integrál na pravé straně přidat znaménko minus.
  (viz příklady)

Stručný zápis:

- $\int\limits_{k} P\,dx + Q\,dy = \int\limits_a^b P\cdot\varphi'(t)dt + Q\cdot\psi'(t)dt = \int\limits_a^b [P\cdot\varphi'(t) + Q\cdot\psi'(t)]\,dt$

kde: $x = \varphi(t) \rightarrow dx = \varphi'(t)dt$
$y = \psi(t) \rightarrow dy = \psi'(t)dt$

$t \in \langle a, b\rangle$

- $\int\limits_{k} P\,dx + Q\,dy + R\,dz = \int\limits_a^b P\cdot\varphi'(t)dt + Q\cdot\psi'(t)dt + R\cdot\rho'(t)dt =$

$= \int\limits_a^b [P\cdot\varphi'(t)dt + Q\cdot\psi'(t) + R\cdot\rho'(t)]\,dt$

kde: $x = \varphi(t) \rightarrow dx = \varphi'(t)dt$
$y = \psi(t) \rightarrow dy = \psi'(t)dt$
$z = \rho(t) \rightarrow dz = \rho'(t)dt$

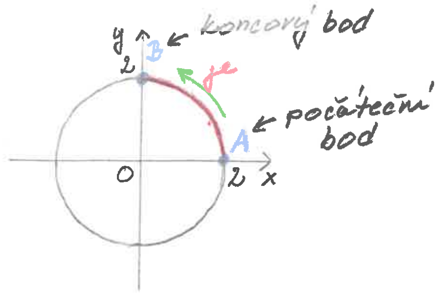
$t \in \langle a, b\rangle$

Př: Vypočtěte křivkové integrály 2. druhu.

1) $\int_{\gamma} y^2dx - x^2dy = *$

γ je orientovaná čtvrtkružnice od bodu $A[2,0]$ do bodu $B[0,2]$.

γ: $x^2+y^2=4$, $x\geq 0$, $y\geq 0$



γ: $x = 2\cos t \;\rightarrow\; dx = -2\sin t\, dt$
 $y = 2\sin t \;\rightarrow\; dy = 2\cos t\, dt$

$t \in \langle 0, \frac{\pi}{2} \rangle$ ... souhlasná (kladná) orientace

$$* = \int_0^{\frac{\pi}{2}} [\underbrace{(2\sin t)^2}_{y^2} \cdot \underbrace{(-2\sin t)}_{dx} - \underbrace{(2\cos t)^2}_{x^2} \cdot \underbrace{2\cos t}_{dy}]\, dt = \int_0^{\frac{\pi}{2}} (-8\sin^3 t - 8\cos^3 t)\, dt =$$

$$= -8\left(\int_0^{\frac{\pi}{2}} \sin^3 t\, dt + \int_0^{\frac{\pi}{2}} \cos^3 t\, dt\right) = -8\left(\int_0^{\frac{\pi}{2}} \underbrace{(1-\cos^2 t)}_{\sin^2 t} \sin t\, dt + \int_0^{\frac{\pi}{2}} \underbrace{(1-\sin^2 t)}_{\cos^2 t} \cos t\, dt\right) =$$

$$= \left| \begin{array}{ll} u = \cos t & \begin{array}{c|c|c} t & 0 & \frac{\pi}{2} \\ \hline u & 1 & 0 \end{array} \\ du = -\sin t\, dt & \\ -du = \sin t\, dt & \end{array} \quad \begin{array}{ll} v = \sin t & \begin{array}{c|c|c} t & 0 & \frac{\pi}{2} \\ \hline v & 0 & 1 \end{array} \\ dv = \cos t\, dt & \end{array} \right| =$$

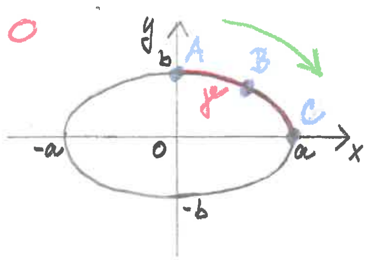
$$= -8\left(\int_0^1 (1-u^2)\, du + \int_0^1 (1-v^2)\, dv\right) = -8\left([u - \tfrac{1}{3}u^3]_0^1 + [v - \tfrac{1}{3}v^3]_0^1\right) =$$

$$= -8\left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right) = -8 \cdot 2 \cdot \left(1 - \tfrac{1}{3}\right) = -16 \cdot \tfrac{2}{3} = -\tfrac{32}{3}$$

2) $\int_{\gamma} (x+y)dx + (x-y)dy = *$

γ je orientovaný oblouk ABC elipsy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

$A[0,b]$, $B[x_B>0, y_B>0]$, $C[a,0]$; $a, b > 0$



γ: $x = a\cos t \;\rightarrow\; dx = -a\sin t\, dt$
 $y = b\sin t \;\rightarrow\; dy = b\cos t\, dt$

$t \in \langle 0, \frac{\pi}{2} \rangle$ ... nesouhlasná (záporná) orientace

$$* = -\int_0^{\frac{\pi}{2}} [(a\cos t + b\sin t)\cdot(-a\sin t) + (a\cos t - b\sin t)\cdot b\cos t]\, dt =$$

$$= -\int_0^{\frac{\pi}{2}} (-a^2\sin t\cos t - ab\sin^2 t + ab\cos^2 t - b^2\sin t\cos t)\, dt =$$

$$= -\int_0^{\frac{\pi}{2}} [ab\cdot\overbrace{(\cos^2 t - \sin^2 t)}^{\cos 2t} - (a^2+b^2)\overbrace{\sin t\cos t}^{\frac{1}{2}\sin 2t}]\, dt =$$

$$= -\int_0^{\frac{\pi}{2}} \left[ab\cos 2t - \tfrac{1}{2}(a^2+b^2)\sin 2t\right]dt = -\left[\tfrac{1}{2}ab\sin 2t + \tfrac{1}{4}(a^2+b^2)\cos 2t\right]_0^{\frac{\pi}{2}} =$$

$$= -\left(\tfrac{1}{2}ab\underbrace{\sin\pi}_{0} + \tfrac{1}{4}(a^2+b^2)\underbrace{\cos\pi}_{-1} - \tfrac{1}{2}ab\underbrace{\sin 0}_{0} - \tfrac{1}{4}(a^2+b^2)\underbrace{\cos 0}_{1}\right) = \tfrac{1}{2}(a^2+b^2)$$

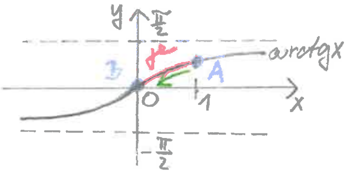
3) $\int_{\varphi} xy\,dx + y^2\,dy = *$

φ je oblouk AB křivky $y = \operatorname{arctg} x$ od bodu $A[1, ?]$ do bodu $B[0, ?]$

φ: $x = t \rightarrow dx = dt$

$y = \operatorname{arctg} t \rightarrow dy = \frac{1}{1+t^2}\,dt$

$t \in \langle 0, 1\rangle$ ... nesouhlasná orientace



$$* = -\int_0^1 \left(t\cdot\operatorname{arctg} t + \operatorname{arctg}^2 t\cdot\frac{1}{1+t^2}\right)dt = -\underbrace{\int_0^1 t\operatorname{arctg} t\,dt}_{A} - \underbrace{\int_0^1 \frac{\operatorname{arctg}^2 t}{1+t^2}\,dt}_{B} = ⊛$$

$$A = \int_0^1 t\operatorname{arctg} t\,dt = \left|\begin{matrix} u = \operatorname{arctg} t & v' = t \\ u' = \frac{1}{1+t^2} & v = \frac{1}{2}t^2 \end{matrix}\right| =$$

$$= \left[\tfrac{1}{2}t^2\operatorname{arctg} t\right]_0^1 - \tfrac{1}{2}\int_0^1 \underbrace{\frac{t^2+1-1}{1+t^2}}_{\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}}dt = \tfrac{1}{2}\left(\left[t^2\operatorname{arctg} t\right]_0^1 - \int_0^1\left(1 - \frac{1}{1+t^2}\right)dt =\right.$$

$$= \tfrac{1}{2}\left[t^2\operatorname{arctg} t - t + \operatorname{arctg} t\right]_0^1 = \left|\operatorname{arctg} 0 = 0,\ \operatorname{arctg} 1 = \tfrac{\pi}{4}\right| =$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{4} - 1 + \tfrac{\pi}{4}\right) = \tfrac{1}{2}\left(\tfrac{\pi}{2} - 1\right) = \tfrac{1}{4}(\pi - 2)$$

$$B = \int_0^1 \frac{\operatorname{arctg}^2 t}{1+t^2}\,dt = \left|\begin{matrix} u = \operatorname{arctg} t \\ du = \frac{1}{1+t^2}\,dt \end{matrix} \quad \begin{array}{c|c|c} t & 0 & 1 \\ u & 0 & \frac{\pi}{4} \end{array}\right| = \int_0^{\frac{\pi}{4}} u^2\,du = \left[\tfrac{1}{3}u^3\right]_0^{\frac{\pi}{4}} =$$

$$= \tfrac{1}{3}\cdot\tfrac{\pi^3}{64} = \tfrac{1}{192}\pi^3$$

$$⊛ = -A - B = -\tfrac{1}{4}(\pi - 2) - \tfrac{1}{192}\pi^3 = \tfrac{1}{192}\left(96 - 48\pi - \pi^3\right)$$

4) $\displaystyle\int_k \frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2-x-y+2z}} = *$

k je orientovaná úsečka AB, $A[1,1,1]$, $B[4,4,4]$

k: $\underline{X = A + t\cdot\overrightarrow{AB}}$ $\qquad \overrightarrow{AB} = (3,3,3)$

$x = 1+3t \quad \to \quad dx = 3\,dt$

$y = 1+3t \quad \to \quad dy = 3\,dt$

$z = 1+3t \quad \to \quad dz = 3\,dt$

$t \in \langle 0, 1\rangle$ ... souhlasná orientace

$$* = \int_0^1 \frac{\overbrace{(1+3t)\cdot 3 + (1+3t)\cdot 3 + (1+3t)\cdot 3}^{9(1+3t)}}{\sqrt{\underbrace{(1+3t)^2+(1+3t)^2+(1+3t)^2}_{3(1+3t)^2}\underbrace{-(1+3t)-(1+3t)+2(1+3t)}_{0}}}\,dt =$$

$$= \int_0^1 \frac{9(1+3t)}{\sqrt{3(1+3t)^2}}\,dt = \int_0^1 \frac{9(1+3t)}{\sqrt{3}\,(1+3t)}\,dt = \frac{9}{\sqrt{3}}\int_0^1 dt = 3\sqrt{3}\,[t]_0^1 = \underline{\underline{3\sqrt{3}}}$$

↳ lze odmocnit, protože pro $t \in \langle 0, 1\rangle$ je $1+3t > 0$

5) $\displaystyle\int_k y\,dx + z\,dy + x\,dz = *$

k je oblouk ABC na průnikové křivce ploch $\overbrace{z = xy}^{z}$, $\overbrace{x^2+y^2=1}^{x,y}$ od bodu $A[1,?,?]$ do bodu $C[-1,?,?]$, je-li $B[?,1,?]$.

k: $x = \cos t \qquad dx = -\sin t\,dt$

$y = \sin t \qquad dy = \cos t\,dt$

$z = \cos t\cdot\sin t = \frac{1}{2}\sin 2t \qquad dz = \frac{1}{2}\cos 2t\cdot 2\,dt = \cos 2t\,dt = (\cos^2 t - \sin^2 t)\,dt$

$t \in \langle 0, \pi\rangle$ ... souhlasná (kladná) orientace

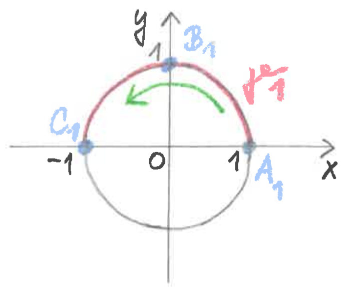
Průmět do roviny (x,y):

A_1 ... průmět bodu A

B_1 ... průmět bodu B

C_1 ... průmět bodu C

k_1 ... průmět křivky k



$$* = \int_0^\pi \Big[\underbrace{\sin t}_{y}\cdot\underbrace{(-\sin t)}_{dx} + \underbrace{\cos t\cdot\sin t}_{z}\cdot\underbrace{\cos t}_{dy} + \underbrace{\cos t}_{x}\,\underbrace{(\cos^2 t - \sin^2 t)}_{dz}\Big]\,dt =$$

$$= \int_0^\pi \Big[\underbrace{-\sin^2 t}_{①} + \underbrace{\sin t\cdot\cos^2 t}_{②} + \underbrace{\cos t\cdot(1-2\sin^2 t)}_{③}\Big]\,dt = (*)$$

$$① = \int_0^\pi (-\sin^2 t)\,dt = -\int_0^\pi \frac{1-\cos 2t}{2}\,dt = -\frac{1}{2}\Big[t - \underbrace{\tfrac{1}{2}\sin 2t}_{0}\Big]_0^\pi = -\frac{\pi}{2}$$

$\sin 0 = \sin 2\pi = 0$

$$② = \int_0^{\pi} \sin t \cdot \cos^2 t \, dt = \left| \begin{array}{l} u = \cos t \\ du = -\sin t \, dt \end{array} \quad \begin{array}{c|c|c} t & 0 & \pi \\ \hline u & 1 & -1 \end{array} \right| =$$

$$= -\int_1^{-1} u^2 du = \int_{-1}^{1} u^2 du = \frac{1}{3}\left[u^3\right]_{-1}^{1} = \frac{1}{3}(1+1) = \frac{2}{3}$$

$$③ = \int_0^{\pi} \cos t \,(1-2\sin^2 t)\, dt = \left| \begin{array}{l} v = \sin t \\ dv = \cos t \, dt \end{array} \quad \begin{array}{c|c|c} t & 0 & \pi \\ \hline v & 0 & 0 \end{array} \right| =$$

$$= \int_0^{0} (1-2v^2)\, dv = 0 \qquad \left(\int_a^a f(x)\,dx = 0 \right)$$

$$(*) = ① + ② + ③ = -\frac{\pi}{2} + \frac{2}{3} + 0 = \underline{\frac{1}{6}(4-3\pi)}$$

6) $\int_{\gamma} x\,dx - 12y\,dy + 18\,dz =$

γ je oblouk AB na průnikové křivce ploch $x+y-1=0$, $9x^2+4y^2-36z=0$ od bodu $A[1,?,?]$ do bodu $B[?,1,?]$

$x+y-1=0 \Rightarrow y = 1-x$ ... rovina kolmá k (x,y)

$9x^2+4y^2-36z=0 \Rightarrow z = \frac{1}{36}(9x^2+4y^2)$ ... eliptický paraboloid, $o = z$

$$\gamma: \; x = t \qquad \rightarrow dx = dt$$

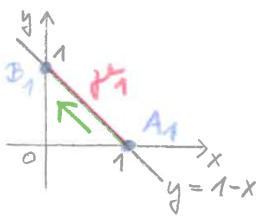
$$y = 1-t \qquad \rightarrow dy = -dt$$

$$z = \frac{1}{36}\left(9t^2+4(1-t)^2\right) = \frac{1}{36}\left(9t^2+4-8t+4t^2\right) = \frac{1}{36}\left(13t^2-8t+4\right) \qquad \rightarrow dz = \frac{1}{36}(26t-8)\,dt = \frac{1}{18}(13t-4)\,dt$$

$t \in \langle 0, 1 \rangle$ ... nesouhlasná orientace

Průmět do roviny (x,y):

(γ je část paraboly v rovině kolmé k (x,y); promítá se do úsečky A_1B_1)



$$* = -\int_0^1 \left[t \cdot 1 - 12(1-t)\cdot(-1) + 18 \cdot \frac{1}{18}(13t-4) \right] dt =$$

$$= -\int_0^1 (t + 12 - 12t + 13t - 4)\, dt = -\int_0^1 (2t+8)\, dt = -\left[t^2+8t\right]_0^1 =$$

$$= -(1+8) = \underline{-9}$$