KŘIVKOVÝ INTEGRÁL PRVNÍHO DRUHU
(VE SKALÁRNÍM POLI)

- přes rovinnou křivku:

$$\int_k f(x,y)\,\underbrace{ds}_{*} = \int_a^b f(\varphi(t),\psi(t))\,\underbrace{\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt}_{*}$$

Křivkový integrál přes křivku k

* délka křivky

($\varphi'(t)$, $\psi'(t)$ - derivace podle proměnné t)

$$k: \left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\} \text{ parametrické vyjádření křivky } k$$

$$t \in \langle a, b \rangle$$

- přes prostorovou křivku:

$$\int_k f(x,y,z)\,\underbrace{ds}_{*} = \int_a^b f(\varphi(t),\psi(t),\rho(t))\,\underbrace{\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2+[\rho'(t)]^2}\,dt}_{*}$$

$$k: \left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \\ z &= \rho(t) \end{aligned}\right\} \text{ parametrické vyjádření křivky } k$$

$$t \in \langle a, b \rangle$$

Pozn.: 1. Není-li křivka k zadána parametricky, musíme její parametrické vyjádření určit.

2. Pro správné určení mezí pro proměnnou t je třeba křivku k nakreslit.

3. Parametrické vyjádření musíme zderivovat; derivujeme vždy podle příslušné proměnné.

derivujeme podle x ← $x = \varphi(t)$
derivujeme podle y ← $y = \psi(t)$ → derivujeme podle t
derivujeme podle z ← $z = \rho(t)$

↓

$$dx = \varphi'(t)\,dt$$
$$dy = \psi'(t)\,dt$$
$$dz = \rho'(t)\,dt$$

PARAMETRICKÉ VYJÁDŘENÍ KŘIVKY

• **Úsečka AB:** $A[a_1, a_2]$, $B[b_1, b_2]$ $\rightarrow$ $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2)$

$$\underline{X = A + t \cdot \overrightarrow{AB}}$$

$$x = a_1 + t \cdot (b_1 - a_1)$$
$$y = a_2 + t \cdot (b_2 - a_2)$$

$$t \in \langle 0, 1 \rangle$$

Vektor $\overrightarrow{AB}$ nelze krátit ani násobit – změnily by se tím meze pro t

($t \in \mathbb{R} \rightarrow$ celá přímka)

• **Kružnice:** $x^2 + y^2 = r^2$ $\rightarrow$

$$x = r\cos t$$
$$y = r\sin t$$
$$t \in \langle 0, 2\pi \rangle \ *$$

• **Elipsa:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\rightarrow$

$$x = a\cos t$$
$$y = b\sin t$$
$$t \in \langle 0, 2\pi \rangle \ *$$

* $t \in \langle 0, 2\pi \rangle \rightarrow$ celá kružnice, resp. elipsa

Pro část kružnice, resp. elipsy, určíme meze pro proměnnou t stejně jako u polárních souřadnic.

• **Přirozená parametrizace:** $y = f(x)$, $x \in \langle a, b \rangle$ $\rightarrow$

$$x = t$$
$$y = f(t)$$
$$t \in \langle a, b \rangle$$

Např.: $y = \ln x$, $x \in \langle 1, e \rangle$ $\rightarrow$

$$x = t$$
$$y = \ln t$$
$$t \in \langle 1, e \rangle$$

Př: Vypočtěte křivkové integrály 1. druhu.

1) $\int\limits_{\gamma} \frac{1}{x-y}\,ds = *$

γ je úsečka AB, $A[0,-2]$, $B[4,0]$

γ: $\underline{X = A + t\cdot\overrightarrow{AB}}$ $\qquad \overrightarrow{AB} = (4,2)$

$x = 4t \quad \to \quad dx = 4\,dt \qquad \left(\varphi(t) = 4t \quad \to \quad \varphi'(t) = 4\right.$

$y = -2+2t \quad \to \quad dy = 2\,dt \qquad \left.\psi(t) = -2+2t \quad \to \quad \psi'(t) = 2\right)$

$t \in \langle 0, 1\rangle$

$$* = \int_0^1 \frac{1}{\underbrace{4t}_{x} - \underbrace{(-2+2t)}_{y}} \sqrt{\underbrace{4^2+2^2}_{[\varphi'(t)]^2+[\psi'(t)]^2}}\,dt = \int_0^1 \frac{1}{2t+2}\cdot\sqrt{20}\,dt = \int_0^1 \frac{2\sqrt{5}}{2(t+1)}\,dt =$$

$$= \sqrt{5}\int_0^1 \frac{1}{t+1}\,dt = \sqrt{5}\,\Big[\ln|t+1|\Big]_0^1 = \sqrt{5}\,(\ln 2 - \underbrace{\ln 1}_{0}) = \underline{\underline{\sqrt{5}\ln 2}}$$

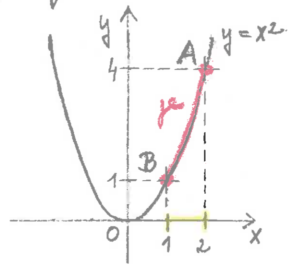
2) $\int\limits_{\gamma} x\,ds = *$

γ: $y = x^2$, $A[2,4]$, $B[1,1]$ ... γ je oblouk $\widehat{AB}$ paraboly $y = x^2$

γ: $x = t \quad \to \quad dx = dt$

$y = t^2 \quad \to \quad dy = 2t\,dt$

$t \in \langle 1, 2\rangle$



$x \in \langle 1, 2\rangle$
$\Downarrow$
$t \in \langle 1, 2\rangle$

$$* = \int_1^2 t\sqrt{1+(2t)^2}\,dt = \int_1^2 t\sqrt{1+4t^2}\,dt = \left| \begin{array}{l} u = 1+4t^2 \\ du = 8t\,dt \\ \frac{1}{8}du = t\,dt \end{array} \quad \begin{array}{c|c|c} t & 1 & 2 \\ \hline u & 5 & 17 \end{array} \right| =$$

$$= \frac{1}{8}\int_5^{17} \sqrt{u}\,du = \frac{1}{8}\int_5^{17} u^{\frac{1}{2}}\,du = \frac{1}{8}\left[\frac{\overset{\sqrt{u^3}}{\overset{\parallel}{u^{\frac{3}{2}}}}}{\frac{3}{2}}\right]_5^{17} = \frac{1}{8}\cdot\frac{2}{3}\,\Big[u\sqrt{u}\Big]_5^{17} =$$

$$= \underline{\underline{\frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right)}}$$

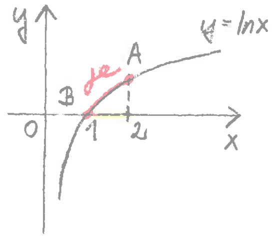
3) $\int\limits_{\gamma} x^2 ds = *$

$\gamma: \; y = \ln x, \; A[2, \ln 2], \; B[1, 0]$ … γ je oblouk AB křivky $y = \ln x$

$\gamma: \; x = t \quad \rightarrow \quad dx = dt$

$y = \ln t \quad \rightarrow \quad dy = \frac{1}{t} dt$

$t \in \langle 1, 2 \rangle$



$x \in \langle 1, 2 \rangle$
$\Downarrow$
$t \in \langle 1, 2 \rangle$

$$* = \int_1^2 t^2 \sqrt{1^2 + \left(\frac{1}{t}\right)^2} dt = \int_1^2 t^2 \sqrt{1 + \frac{1}{t^2}} dt = \int_1^2 t^2 \sqrt{\frac{t^2+1}{t^2}} dt = \int_1^2 t^2 \cdot \frac{\sqrt{t^2+1}}{t} dt =$$

$$= \int_1^2 t \sqrt{t^2+1} \, dt = \left| \begin{array}{l} u = t^2 + 1 \\ du = 2t \, dt \\ \frac{1}{2} du = dt \end{array} \quad \begin{array}{c|c|c} t & 1 & 2 \\ \hline u & 2 & 5 \end{array} \right| = \frac{1}{2} \int_2^5 \underbrace{\sqrt{u}}_{= u^{\frac{1}{2}}} du = \frac{1}{2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_2^5 =$$

$\sqrt{u^3} = u\sqrt{u}$

$$= \frac{1}{3} \left[u \sqrt{u} \right]_2^5 = \frac{1}{3} (5\sqrt{5} - 2\sqrt{2})$$

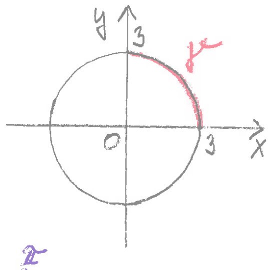
4) $\int\limits_{\gamma} x^2 y \, ds = *$

$\gamma: \; x^2 + y^2 = 9, \; x \geq 0, \; y \geq 0$ … γ je čtvrtkružnice se středem v počátku a poloměrem 3

$\gamma: \; x = 3 \cos t \quad \rightarrow \quad dx = -3 \sin t \, dt$

$y = 3 \sin t \quad \rightarrow \quad dy = 3 \cos t \, dt$

$t \in \langle 0, \frac{\pi}{2} \rangle$



$$* = \int_0^{\frac{\pi}{2}} (3 \cos t)^2 \cdot 3 \sin t \cdot \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} \, dt = \int_0^{\frac{\pi}{2}} 9 \cos^2 t \cdot 3 \sin t \cdot \sqrt{9 \sin^2 t + 9 \cos^2 t} \, dt =$$

$$= \int_0^{\frac{\pi}{2}} 27 \cos^2 t \cdot \sin t \cdot \sqrt{9 \underbrace{(\sin^2 t + \cos^2 t)}_{1}} \, dt = 27 \cdot 3 \int_0^{\frac{\pi}{2}} \cos^2 t \cdot \sin t \, dt =$$

$$= \left| \begin{array}{l} u = \cos t \\ du = -\sin t \, dt \\ -du = \sin t \, dt \end{array} \quad \begin{array}{c|c|c} t & 0 & \frac{\pi}{2} \\ \hline u & 1 & 0 \end{array} \right| = -81 \int_1^0 u^2 du = 81 \int_0^1 u^2 du =$$

$$= 81 \left[\frac{u^3}{3} \right]_0^1 = 27 (1^3 - 0^3) = 27$$

5) $\int_\gamma (x-y)\,ds = *$

$\gamma:\ x^2+y^2-4x=0$ … γ je kružnice se středem v bodě $[2,0]$ a poloměrem 2

$(x-2)^2+y^2=4$

$\gamma:\ x=2\cos t+2 \ \rightarrow\ dx=-2\sin t\,dt$

$\quad y=2\sin t \ \rightarrow\ dy=2\cos t\,dt$

$t\in\langle 0,2\pi\rangle$

Pozn: $(x-m)^2+(y-n)^2=r^2$ … kružnice se středem v bodě $[m,n]$ a poloměrem r

$\downarrow$

$x=r\cos t + m$

$y=r\sin t + n$ $\longrightarrow$ posune střed kružnice do počátku

$$* = \int_0^{2\pi}(2\cos t+2-2\sin t)\sqrt{\underbrace{(-2\sin t)^2+(2\cos t)^2}_{4\sin^2 t+4\cos^2 t=4\underbrace{(\sin^2 t+\cos^2 t)}_{1}=4}}\,dt=\int_0^{2\pi}2(1+\cos t-\sin t)\cdot\underbrace{\sqrt{4}}_{2}\,dt=$$

$$=4\int_0^{2\pi}(1+\cos t-\sin t)\,dt=4\left[t+\sin t+\cos t\right]_0^{2\pi}=$$

$$=4(2\pi+\underbrace{\sin 2\pi}_{0}+\underbrace{\cos 2\pi}_{1}-0-\underbrace{\sin 0}_{0}-\underbrace{\cos 0}_{1})=4\cdot(2\pi+1-1)=\underline{\underline{8\pi}}$$

6) $\int_\gamma xy\,ds = *$

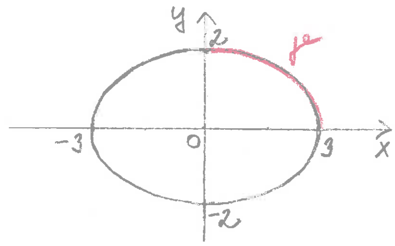
$\gamma:\ 4x^2+9y^2=36,\ x\geq 0,\ y\geq 0$ … γ je čtvrtelipsa se středem v počátku

$\frac{x^2}{9}+\frac{y^2}{4}=1 \ \Rightarrow\ a=3,\ b=2$ … délky poloos

$\gamma:\ x=3\cos t \ \rightarrow\ dx=-3\sin t\,dt$

$\quad y=2\sin t \ \rightarrow\ dy=2\cos t\,dt$

$t\in\langle 0,\frac{\pi}{2}\rangle$



$$* = \int_0^{\frac{\pi}{2}}3\cos t\cdot 2\sin t\cdot\sqrt{(-3\sin t)^2+(2\cos t)^2}\,dt = 6\int_0^{\frac{\pi}{2}}\sin t\cos t\cdot\sqrt{9\sin^2 t+4\cos^2 t}\,dt=$$

$$=\left|\begin{array}{l} u=9\sin^2 t+4\cos^2 t \\ du=[9\cdot 2\sin t\cos t+4\cdot 2\cos t\cdot(-\sin t)]\,dt=\sin t\cos t\cdot(18-8)\,dt= \\ \quad =10\sin t\cos t\,dt \\ \frac{1}{10}du=\sin t\cos t\,dt \end{array}\quad \begin{array}{c|c|c} t & 0 & \frac{\pi}{2} \\ \hline u & 4 & 9 \end{array}\right|=$$

$$=6\cdot\frac{1}{10}\int_4^9\overset{=u^{\frac{1}{2}}}{\sqrt{u}}\,du=\frac{3}{5}\left[\frac{\overset{=\sqrt{u^3}=u\sqrt{u}}{u^{\frac{3}{2}}}}{\frac{3}{2}}\right]_4^9=\frac{3}{5}\cdot\frac{2}{3}\left[u\sqrt{u}\right]_4^9=\frac{2}{5}(9\sqrt{9}-4\sqrt{4})=\frac{2}{5}(9\cdot 3-4\cdot 2)=$$

$$=\frac{2}{5}(27-8)=\frac{2}{5}\cdot 19=\underline{\underline{\frac{38}{5}}}$$

7) $\int\limits_{\gamma} \frac{z^2}{x^2+y^2}\,ds = *$

γ: $\left.\begin{array}{l} x = a\cos t \\ y = a\sin t \\ z = a t \end{array}\right\}$ šroubovice $\begin{array}{l} \to \quad dx = -a\sin t\,dt \\ \to \quad dy = a\cos t\,dt \\ \to \quad dz = a\,dt \end{array}$

$t \in \langle 0, 2\pi\rangle, \; a > 0$

$$* = \int_0^{2\pi} \frac{(at)^2}{\underbrace{(a\cos t)^2 + (a\sin t)^2}_{a^2\cos^2 t + a^2\sin^2 t = a^2}} \sqrt{\underbrace{(-a\sin t)^2 + (a\cos t)^2}_{a^2\sin^2 t + a^2\cos^2 t = a^2} + a^2}\; dt = \int_0^{2\pi} \frac{a^2 t^2}{a^2} \underbrace{\sqrt{2a^2}}_{a\sqrt{2}}\, dt =$$

$$= a\sqrt{2}\int_0^{2\pi} t^2\,dt = a\sqrt{2}\left[\frac{t^3}{3}\right]_0^{2\pi} = \frac{1}{3} a\sqrt{2}\cdot 8\pi^3 = \underline{\underline{\frac{8\sqrt{2}}{3}\pi^3 a}}$$

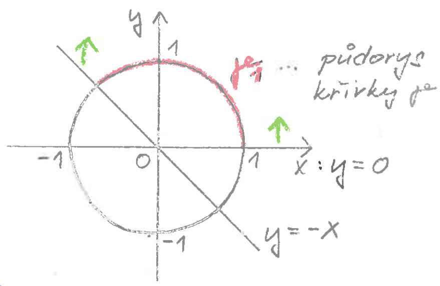
8) $\int\limits_{\gamma} 2(z-y^2)xy\,ds = *$

γ je průniková křivka ploch $\overbrace{x^2+y^2=1}^{x,y\ \Uparrow}$, $\overbrace{z = x^2}^{z\ \Uparrow}$ pro $\overbrace{y \geq 0,\ y \geq -x}^{t\ \Uparrow}$.

$x^2+y^2=1$ ... rotační válcová plocha o poloměru 1, $o = z$

$z = x^2$ ... parabolická válcová plocha

Průmět do roviny (x,y):



γ: $\begin{array}{l} x = \cos t \\ y = \sin t \\ z = \cos^2 t \end{array}$ $\begin{array}{l} \to \quad dx = -\sin t\,dt \\ \to \quad dy = \cos t\,dt \\ \to \quad dz = -\underbrace{2\cos t\sin t}_{\sin 2t}\,dt = -\sin 2t\,dt \end{array}$

$t \in \langle 0, \frac{3}{4}\pi\rangle$

$$* = \int_0^{\frac{3}{4}\pi} \underbrace{2\overbrace{(\cos^2 t - \sin^2 t)}^{\cos 2t}\cos t\sin t}_{\sin 2t} \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + \sin^2 2t}\; dt =$$

$$= \int_0^{\frac{3}{4}\pi} \sin 2t \cdot \cos 2t \cdot \sqrt{1+\sin^2 2t}\; dt = \left| \begin{array}{l} u = 1+\sin^2 2t \\ du = 2\sin 2t\cos 2t\cdot 2\,dt \\ \frac{1}{4}du = \sin 2t\cos 2t\,dt \end{array} \quad \begin{array}{c|c|c} t & 0 & \frac{3}{4}\pi \\ \hline u & 1 & 2 \circledast \end{array} \right| =$$

$\circledast$ $\sin^2(2\cdot\frac{3}{4}\pi) = \sin^2\frac{3}{2}\pi = (-1)^2 = 1$

$$= \frac{1}{4}\int_1^2 \sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}\left[u\sqrt{u}\right]_1^2 = \underline{\underline{\frac{1}{6}(2\sqrt{2}-1)}}$$