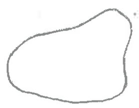
GREENOVA VĚTA

$$\oint_{\gamma} P\,dx + Q\,dy = \iint_{D} (Q_x - P_y)\,dx\,dy$$

γ je jednoduchá, uzavřená, kladně orientovaná křivka, která ohraničuje množinu D.

Pozn.: Je-li křivka γ orientovaná záporně, musíme před dvojný integrál dát znaménko minus.

Pozn.: • Jednoduchá křivka – nekříží se



ano

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• Uzavřená křivka – začíná a končí ve stejném bodě

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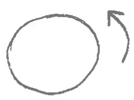
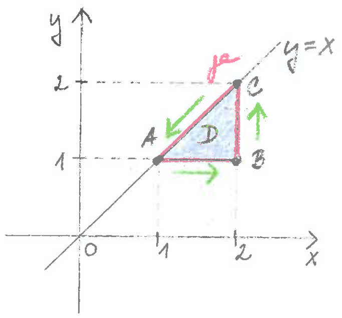
• Kladná orientace – proti směru chodu hodinových ručiček



Záporná orientace – po směru chodu hodinových ručiček

1) $\displaystyle\oint_{k} \frac{1}{y}dx - \frac{1}{x}dy = *$

k je kladně orientovaná hranice trojúhelníku ABC,
$A[1,1]$, $B[2,1]$, $C[2,2]$



$$D: \quad 1 \le x \le 2$$
$$\quad\;\; 1 \le y \le x$$

$$P = \frac{1}{y} \quad \rightarrow \quad P_y = -\frac{1}{y^2}$$

$$Q = -\frac{1}{x} \quad \rightarrow \quad Q_x = \frac{1}{x^2}$$

$$= y^{-2} \rightarrow \frac{y^{-1}}{-1} = -\frac{1}{y}$$

$$* = \iint_D \left[\frac{1}{x^2} - \left(-\frac{1}{y^2}\right)\right] dx\,dy = \int_1^2 \left[\int_1^x \left(\frac{1}{x^2} + \frac{1}{y^2}\right) dy\right] dx = \int_1^2 \left[\frac{y}{x^2} - \frac{1}{y}\right]_1^x dx =$$

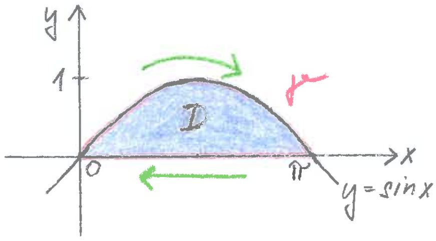
$$= \int_1^2 \left(\frac{x}{x^2} - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{1}\right) dx = \int_1^2 \left(\frac{1}{x} - \frac{1}{x} - \frac{1}{x^2} + 1\right) dx = \int_1^2 \left(1 - \frac{1}{x^2}\right) dx =$$

$$= -x^{-2} = -\frac{x^{-1}}{-1} = \frac{1}{x}$$

$$= \left[x + \frac{1}{x}\right]_1^2 = 2 + \frac{1}{2} - 1 - \frac{1}{1} = \underline{\underline{\frac{1}{2}}}$$

2) $\oint\limits_{\gamma} (x+y)^2 dx - (x-y)^2 dy = *$

γ je uzavřená záporně orientovaná křivka tvořená sinusoidou $y = \sin x$ a úsečkou na ose x pro $0 \leq x \leq \pi$.



$D: \; 0 \leq x \leq \pi$
$\quad\;\; 0 \leq y \leq \sin x$

$P = (x+y)^2 \quad \rightarrow \quad P_y = 2(x+y)$

$Q = -(x-y)^2 \quad \rightarrow \quad Q_x = -2(x-y)$

záporná orientace

$$* = -\iint\limits_D [-2(x-y) - 2(x+y)]\,dx\,dy = -\iint\limits_D (-2x+2y-2x-2y)\,dx\,dy =$$

$$= 4\iint\limits_D x\,dx\,dy = 4\int_0^{\pi}\left[\int_0^{\sin x} x\,dy\right]dx = 4\int_0^{\pi} [xy]_0^{\sin x}\,dx = 4\int_0^{\pi} x \sin x\,dx =$$

$$= \left| \begin{array}{ll} u = x & v' = \sin x \\ u' = 1 & v = -\cos x \end{array} \right| = 4\left([-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x\,dx\right) =$$

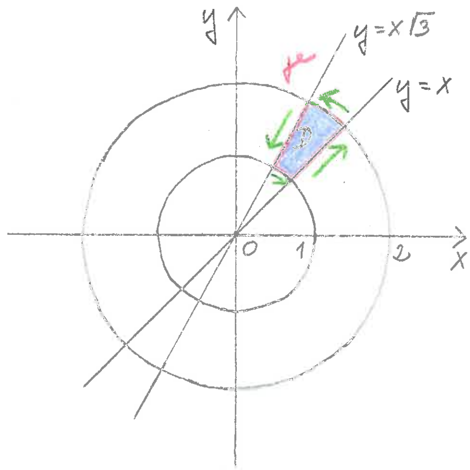
$$= 4[-x\cos x + \sin x]_0^{\pi} = 4(-\pi \underbrace{\cos\pi}_{-1} + \underbrace{\sin\pi}_{0} + 0\cdot\underbrace{\cos 0}_{1} - \underbrace{\sin 0}_{0}) =$$

$$= 4(-\pi\cdot(-1)) = \underline{\underline{4\pi}}$$

3) $\int\limits_{\mu} \frac{1}{x} \operatorname{arctg} \frac{y}{x}\, dx + \frac{2}{y} \operatorname{arctg} \frac{x}{y}\, dy = *$

μ je kladně orientovaná hranice množiny D,

$D:\ 1 \leq x^2+y^2 \leq 4\ ,\quad x \leq y \leq x\sqrt{3}$



→ Transformace do polárních souřadnic:

$1 \leq r \leq 2$

$\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{3}$

$P = \frac{1}{x} \operatorname{arctg} \frac{y}{x} \quad \rightarrow \quad P_y = \frac{1}{x} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{x^2} \cdot \frac{1}{\frac{x^2+y^2}{x^2}} = \frac{1}{x^2+y^2}$

$Q = \frac{2}{y} \operatorname{arctg} \frac{x}{y} \quad \rightarrow \quad Q_x = \frac{2}{y} \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{2}{y^2} \cdot \frac{1}{\frac{y^2+x^2}{y^2}} = \frac{2}{x^2+y^2}$

$$* = \iint\limits_D \left(\frac{2}{x^2+y^2} - \frac{1}{x^2+y^2} \right) dx\,dy = \iint\limits_D \frac{1}{x^2+y^2}\, dx\,dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[\int_1^2 \frac{1}{r^2} \cdot r\, dr \right] d\varphi =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[\int_1^2 \frac{1}{r}\, dr \right] d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[\ln|r| \right]_1^2 d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\ln 2 - \underbrace{\ln 1}_{0})\, d\varphi = \ln 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi =$$

$$= \ln 2 \cdot \left[\varphi \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \ln 2 \cdot \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \underline{\underline{\frac{\pi}{12} \ln 2}}$$