APLIKACE TROJNÉHO INTEGRÁLU

Př: Vypočtete objem tělesa W:

1, $W: 2x+6y+z=6, \; x=0, \; y=0, \; z=0$

2, $W: x^2+y^2 \leq 1, \; x+y+z \leq 6, \; z \geq 0$

3, $W: 1 \leq x^2+y^2+z^2 \leq 4, \; z \geq 0$

4, $W: x=0, \; y=0, \; z=0, \; x+y=1, \; z=x^2+y^2+1$

Př: Vypočtěte hmotnost tělesa W s hustotou $\sigma(x,y,z)$:

1, $W: z=1-y^2, \; z=0, \; x=0, \; x=\pi; \; \sigma(x,y,z)=\sin x$

APLIKACE TROJNÉHO INTEGRÁLU

Př: Vypočtěte objem tělesa W.

1) $W: 2x+6y+z=6, \; x=0, \; y=0, \; z=0$

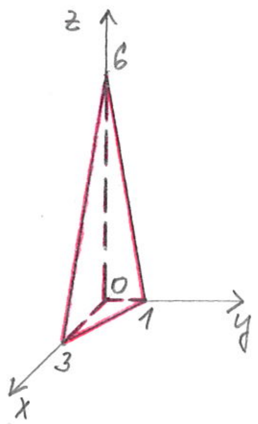
$\to \frac{x}{3}+y+\frac{z}{6}=1$

$\to z=6-2x-6y$

Průsečnice s (x,y):

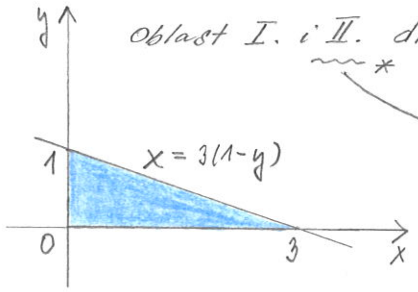
$z=0 \Rightarrow 2x+6y=6$

$x=3-3y=3(1-y)$



Průmět do (x,y):

Oblast I. i II. druhu *



$$\begin{cases} 0 \le y \le 1 \\ 0 \le x \le 3(1-y) \\ 0 \le z \le 6-2x-6y \end{cases}$$

(* jednodušší výpočet)

$$V=\iiint\limits_W dx\,dy\,dz=\int_0^1\left[\int_0^{3(1-y)}\left[\int_0^{6-2x-6y} dz\right]dx\right]dy=\int_0^1\left[\int_0^{3(1-y)}\left[z\right]_0^{6-2x-6y}dx\right]dy=$$

$$=\int_0^1\left[\int_0^{3(1-y)}(6-2x-6y)\,dx\right]dy=\int_0^1\left[6x-x^2-6xy\right]_0^{3(1-y)}dy=$$

$$=\int_0^1\left[18(1-y)-9(1-y)^2-18y(1-y)\right]dy=\left|\begin{matrix}\text{vytkneme 9 a poté}\\ \text{roznásobíme ⊛}\end{matrix}\right|=$$

$$=9\int_0^1(2-2y-1+2y-y^2-2y+2y^2)\,dy=9\int_0^1(y^2-2y+1)\,dy=$$

$$=9\int_0^1(y-1)^2dy=9\left[\frac{(y-1)^3}{3}\right]_0^1=3\left[(y-1)^3\right]_0^1=3(0-(-1)^3)=\underline{\underline{3}}$$

⊛ nebo vytkneme $9(1-y)$:

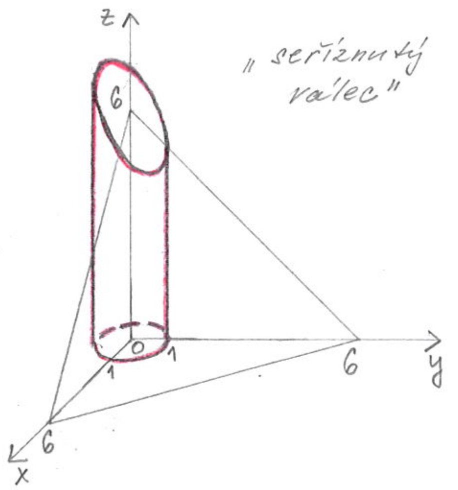
$$18(1-y)-9(1-y)^2-18y(1-y)=9(1-y)(2-1+y-2y)=9(1-y)(1-y)=$$

$$=9(1-y)^2=9(y-1)^2$$

2) $W: x^2+y^2 \le 1, \quad x+y+z \le 6, \quad z \ge 0$

$x^2+y^2=1$ ... rotační válcová plocha o poloměru 1, $o = z$

$x+y+z=6$ ... rovina $\rightarrow$ $z = 6-x-y = 6-r\cos\varphi - r\sin\varphi$ *



„seříznutý válec" $\rightarrow$ transformace do cylindrických souřadnic:

$$0 \le r \le 1$$
$$0 \le \varphi \le 2\pi$$
$$0 \le z \le 6-r\cos\varphi - r\sin\varphi \ *$$

$$V = \iiint_W dx\,dy\,dz = \int_0^1 \left[\int_0^{2\pi} \left[\int_0^{6-r\cos\varphi - r\sin\varphi} r \, dz \right] d\varphi \right] dr =$$

Jakobián!

$$= \int_0^1 \left[\int_0^{2\pi} r \, [z]_0^{6-r\cos\varphi - r\sin\varphi} \, d\varphi \right] dr =$$

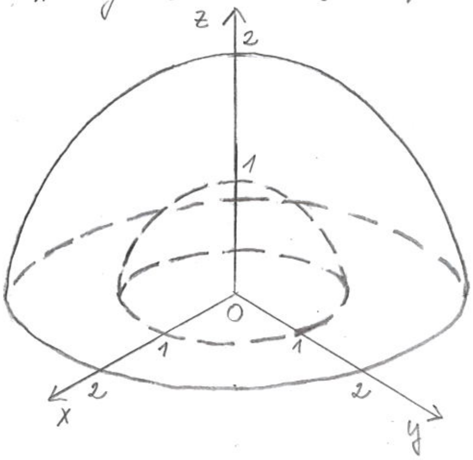
$$= \int_0^1 \left[\int_0^{2\pi} r(6 - r\cos\varphi - r\sin\varphi)\, d\varphi \right] dr = \int_0^1 r \left[6\varphi - r\sin\varphi + r\cos\varphi \right]_0^{2\pi} dr =$$

$$= \int_0^1 r \, (12\pi - r\underbrace{\sin 2\pi}_{0} + r\underbrace{\cos 2\pi}_{1} - 0 + r\underbrace{\sin 0}_{0} - r\underbrace{\cos 0}_{1})\, dr =$$

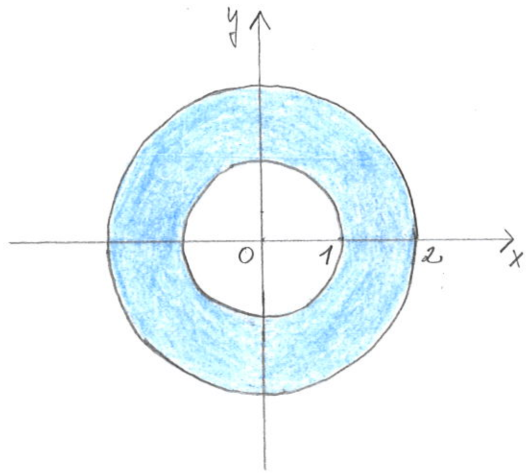
$$= \int_0^1 r\,(12\pi + r - r)\, dr = 12\pi \int_0^1 r\, dr = 12\pi \left[\frac{r^2}{2} \right]_0^1 = \underline{\underline{6\pi}}$$

3) $W: \ 1 \le x^2+y^2+z^2 \le 4 \ , \quad z \ge 0$

$$\left.\begin{array}{l} x^2+y^2+z^2=1 \\ x^2+y^2+z^2=4 \end{array}\right\}$$ kulové plochy se středem v počátku o poloměrech 1 a 2



Řez rovinou (x,y):



Transformace do sférických souřadnic:

$$1 \le r \le 2$$
$$0 \le \varphi \le 2\pi$$
$$0 \le \vartheta \le \frac{\pi}{2}$$

$$V = \iiint_W dx\,dy\,dz = \int_0^{2\pi}\left[\int_1^2\left[\int_0^{\frac{\pi}{2}} \underbrace{r^2\cos\vartheta}_{\text{Jakobián!}}\,d\vartheta\right]dr\right]d\varphi = \int_0^{2\pi}\left[\int_1^2 r^2\left[\sin\vartheta\right]_0^{\frac{\pi}{2}}dr\right]d\varphi =$$

$$= \int_0^{2\pi}\left[\int_1^2 r^2\left(\underbrace{\sin\tfrac{\pi}{2}}_{1} - \underbrace{\sin 0}_{0}\right)dr\right]d\varphi = \int_0^{2\pi}\left[\frac{r^3}{3}\right]_1^2 d\varphi = \int_0^{2\pi}\left(\frac{8}{3}-\frac{1}{3}\right)d\varphi =$$

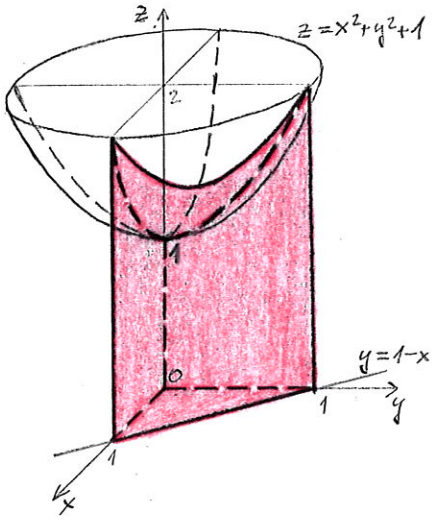
$$= \frac{7}{3}\int_0^{2\pi} d\varphi = \frac{7}{3}\left[\varphi\right]_0^{2\pi} = \frac{7}{3}\cdot 2\pi = \underline{\underline{\frac{14}{3}\pi}}$$

4) $W: x=0,\ y=0,\ z=0,\ x+y=1,\ z=x^2+y^2+1$

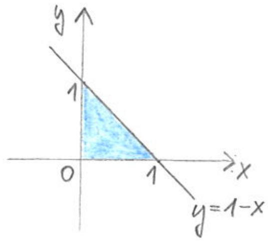
$\to$ rovina kolmá k (x,y)

$\to y = 1-x$

$\to$ rotační eliptický paraboloid osa $o=z$, vrchol $V[0,0,1]$



Průmět do (x,y):



$$0 \le x \le 1$$
$$0 \le y \le 1-x$$
$$0 \le z \le x^2+y^2+1$$

$$V = \iiint_W dx\,dy\,dz = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{x^2+y^2+1} dz \right] dy \right] dx = \int_0^1 \left[\int_0^{1-x} [z]_0^{x^2+y^2+1} dy \right] dx =$$

$$= \int_0^1 \left[\int_0^{1-x} (x^2+y^2+1)\, dy \right] dx = \int_0^1 \left[x^2 y + \frac{1}{3}y^3 + y \right]_0^{1-x} dx =$$

$$= \int_0^1 \left[x^2(1-x) + \frac{1}{3}\underbrace{(1-x)^3}_{1-3x+3x^2-x^3} + 1-x \right] dx = \int_0^1 \left(x^2 - x^3 + \frac{1}{3} - x + x^2 - \frac{1}{3}x^3 + 1 - x \right) dx =$$

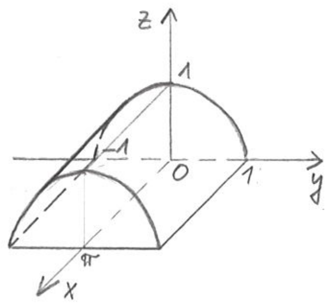
$$= \int_0^1 \left(\frac{4}{3} - 2x + 2x^2 - \frac{4}{3}x^3 \right) dx = \left[\frac{4}{3}x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 \right]_0^1 =$$

$$= \frac{4}{3} - 1 + \frac{2}{3} - \frac{1}{3} = \frac{5}{3} - 1 = \underline{\underline{\frac{2}{3}}}$$

Př: Vypočtěte hmotnost tělesa W s hustotou $\sigma(x,y,z)$.

1) $W: z = 1-y^2,\ z=0,\ x=0,\ x=\pi;\quad \sigma(x,y,z) = \sin x$

parabolická válcová plocha



meze pro y: $\left.\begin{array}{l} z = 1-y^2 \\ z = 0 \end{array}\right\} \Rightarrow 1-y^2 = 0$

$y^2 = 1$

$y = \pm 1$

$0 \leq x \leq \pi$

$-1 \leq y \leq 1$

$0 \leq z \leq 1-y^2$

$$m = \iiint_W \sigma(x,y,z)\,dx\,dy\,dz = \iiint_W \sin x\,dx\,dy\,dz = \int_0^{\pi}\Big[\int_{-1}^{1}\Big[\int_0^{1-y^2} \sin x\,dz\Big]dy\Big]dx =$$

$$= \int_0^{\pi}\Big[\int_{-1}^{1} \sin x \cdot [z]_0^{1-y^2}\,dy\Big]dx = \int_0^{\pi}\Big[\int_{-1}^{1}(1-y^2)\sin x\,dy\Big]dx =$$

$$= \int_0^{\pi} \sin x \cdot \Big[y - \tfrac{1}{3}y^3\Big]_{-1}^{1} dx = \int_0^{\pi} \sin x \cdot \Big(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\Big)\,dx =$$

$$= \frac{4}{3}\int_0^{\pi} \sin x\,dx = \frac{4}{3}\big[-\cos x\big]_0^{\pi} = \frac{4}{3}\big(-\underbrace{\cos\pi}_{-1} + \underbrace{\cos 0}_{1}\big) =$$

$$= \frac{4}{3}\cdot 2 = \frac{8}{3}$$