TRANSFORMACE TROJNÉHO INTEGRÁLU
DO SFÉRICKÝCH SOUŘADNIC

Transformaci do sférických souřadnic využíváme, je-li integrační obor ohraničen kulovou plochou.

Transformační rovnice:

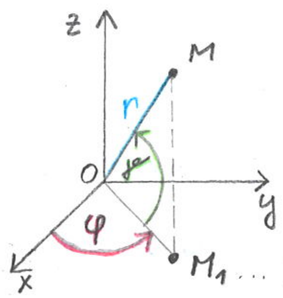
$x = r\cos\varphi\cos\mu$

$y = r\sin\varphi\cos\mu$

$z = r\sin\mu$

$J = r^2\cos\mu$ *

$\rightarrow x^2+y^2+z^2 = r^2$ **



$M[x,y,z] \rightarrow M[r,\varphi,\mu]$

$$\rightarrow \iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W'} f(r\cos\varphi\cos\mu,\ r\sin\varphi\cos\mu,\ r\sin\mu)\cdot r^2\cos\mu\; dr\,d\varphi\,d\mu$$

Pro meze pro proměnné r, φ, μ musí platit:

1. Rozsah mezí pro proměnnou φ musí být maximálně 2π.

2. Meze pro proměnnou r nesmí být záporné (proměnná r určuje vzdálenost od počátku).

Meze pro proměnné r a φ tedy určujeme obdobně jako u polárních souřadnic – často využíváme průmět tělesa W do roviny (x,y).

3. Obě meze pro proměnnou μ musí být z intervalu $\langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$.

Např.: $W: x^2+y^2+z^2 \le a^2 \longrightarrow W': \langle 0,a\rangle \times \langle 0,2\pi\rangle \times \langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$

koule se středem v počátku a poloměrem a

trojrozměrný interval (kvádr)

$0 \le r \le a$

$0 \le \varphi \le 2\pi$

$-\frac{\pi}{2} \le \mu \le \frac{\pi}{2}$

$$* \quad J = \begin{vmatrix} (r\cos\varphi\cos\psi)'_r & (r\cos\varphi\cos\psi)'_\varphi & (r\cos\varphi\cos\psi)'_\psi \\ (r\sin\varphi\cos\psi)'_r & (r\sin\varphi\cos\psi)'_\varphi & (r\sin\varphi\cos\psi)'_\psi \\ (r\sin\psi)'_r & (r\sin\psi)'_\varphi & (r\sin\psi)'_\psi \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\varphi\cos\psi & -r\sin\varphi\cos\psi & -r\cos\varphi\sin\psi \\ \sin\varphi\cos\psi & r\cos\varphi\cos\psi & -r\sin\varphi\sin\psi \\ \sin\psi & 0 & r\cos\psi \end{vmatrix} =$$

$$= \sin\psi \cdot \begin{vmatrix} -r\sin\varphi\cos\psi & -r\cos\varphi\sin\psi \\ r\cos\varphi\cos\psi & -r\sin\varphi\sin\psi \end{vmatrix} + r\cos\psi \cdot \begin{vmatrix} \cos\varphi\cos\psi & -r\sin\varphi\cos\psi \\ \sin\varphi\cos\psi & r\cos\varphi\cos\psi \end{vmatrix} =$$

$$= \sin\psi \cdot (r^2\sin^2\varphi\sin\psi\cos\psi + r^2\cos^2\varphi\sin\psi\cos\psi) +$$

$$+ r\cos\psi\,(r\cos^2\varphi\cos^2\psi + r\sin^2\varphi\cos^2\psi) =$$

$$= r^2\sin^2\psi\cos\psi \cdot (\underbrace{\sin^2\varphi + \cos^2\varphi}_{1}) + r^2\cos^3\psi\,(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) =$$

$$= r^2\cos\psi\,(\underbrace{\sin^2\psi + \cos^2\psi}_{1}) = r^2\cos\psi$$

$$** \quad x^2 + y^2 + z^2 = r^2\cos^2\varphi\cos^2\psi + r^2\sin^2\varphi\cos^2\psi + r^2\sin^2\psi =$$

$$= r^2\cos^2\psi \cdot (\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) + r^2\sin^2\psi =$$

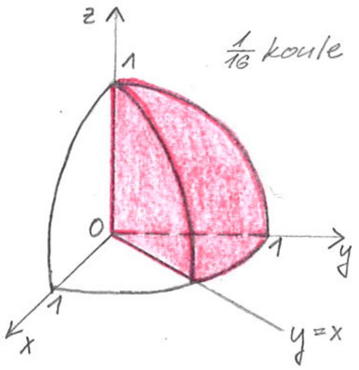
$$= r^2(\underbrace{\cos^2\psi + \sin^2\psi}_{1}) = r^2$$

Př: Pomocí transformace do sférických souřadnic vypočtěte následující integrály.

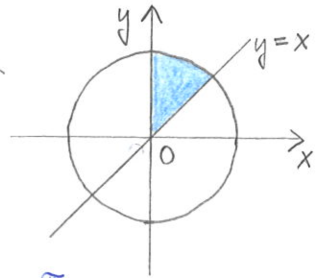
1, $\iiint_W \underbrace{\sqrt{x^2+y^2+z^2}}_{\sqrt{r^2}\,=\,r}\,dxdydz =$

$W: \underbrace{0 \le x \le y}_{x \ge 0 \,\wedge\, y \ge x},\ z \ge 0,\ x^2+y^2+z^2 \le 1$

koule se středem v počátku a poloměrem 1



Průmět do (x,y):



$0 \le r \le 1$
$\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$
$0 \le \psi \le \frac{\pi}{2}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\int_0^1 \left[\int_0^{\frac{\pi}{2}} r \cdot \overbrace{r^2 \cos\psi}^{\text{Jakobián}}\, d\psi \right] dr \right] d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\int_0^1 r^3 \left[\sin\psi\right]_0^{\frac{\pi}{2}} dr \right] d\varphi =$$

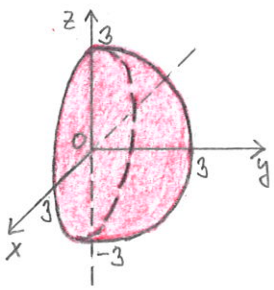
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\int_0^1 r^3 (\underbrace{\sin\tfrac{\pi}{2}}_{1} - \underbrace{\sin 0}_{0}) dr \right] d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^1 d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\, d\varphi = \frac{1}{4}\left[\varphi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{4} \cdot \frac{\pi}{4} = \underline{\underline{\frac{1}{16}\pi}}$$

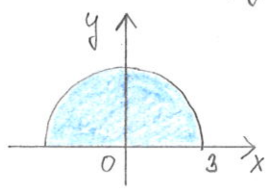
2) $\iiint_W y\, dxdydz =$

$W: x^2+y^2+z^2 \le 9,\ \ y \ge 0$

1/2 koule se středem v počátku a poloměrem 3



Průmět do (x,y):



$0 \le r \le 3$
$0 \le \varphi \le \pi$
$-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^3 \left[\int_0^{\pi} r\sin\varphi\cos\psi \cdot r^2\cos\psi\, d\varphi \right] dr \right] d\psi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^3 r^3\cos^2\psi \left[-\cos\varphi\right]_0^{\pi} dr \right] d\psi =$$

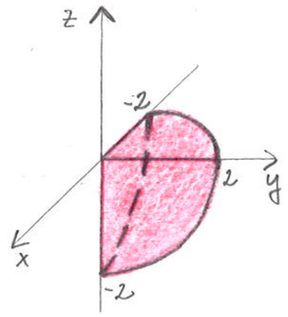
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^3 r^3\cos^2\psi\,(\underbrace{-\cos\pi}_{-1} + \underbrace{\cos 0}_{1})\, dr \right] d\psi = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^3 \cos^2\psi\, d\psi = \frac{81}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\psi)\, d\psi =$$

$$= \frac{81}{4}\left[\psi + \frac{1}{2}\sin 2\psi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{81}{4}\left(\frac{\pi}{2} + \frac{1}{2}\underbrace{\sin\pi}_{0} - \left(-\frac{\pi}{2}\right) - \frac{1}{2}\underbrace{\sin(-\pi)}_{0}\right) = \underline{\underline{\frac{81}{4}\pi}}$$

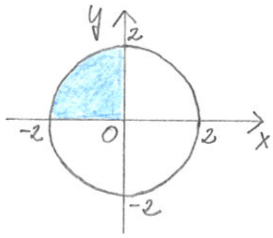
3, $\iiint\limits_W xyz\,dx\,dy\,dz =$

$W: x^2+y^2+z^2 \le 4,\ x \le 0,\ y \ge 0,\ z \le 0$

1/8 koule se středem v počátku a poloměrem 2



Průmět do (x,y):



$$0 \le r \le 2$$
$$\frac{\pi}{2} \le \varphi \le \pi$$
$$-\frac{\pi}{2} \le \psi \le 0$$

$$= \int_0^2 \left[\int_{\frac{\pi}{2}}^{\pi} \left[\int_{-\frac{\pi}{2}}^{0} \underbrace{r\cos\varphi\cos\psi}_{x} \cdot \underbrace{r\sin\varphi\cos\psi}_{y} \cdot \underbrace{r\sin\psi}_{z} \cdot \underbrace{r^2\cos\psi}_{J}\, d\psi \right] d\varphi \right] dr =$$

$$= \int_0^2 \left[\int_{\frac{\pi}{2}}^{\pi} \left[\int_{-\frac{\pi}{2}}^{0} r^5 \sin\varphi\cos\varphi\sin\psi\cos^3\psi\, d\psi \right] d\varphi \right] dr = \left| \begin{array}{l} t = \cos\psi \\ dt = -\sin\psi \end{array} \quad \begin{array}{c|c|c} \psi & -\frac{\pi}{2} & 0 \\ \hline t & 0 & 1 \end{array} \right| =$$

$$= \int_0^2 \left[\int_{\frac{\pi}{2}}^{\pi} \left[\int_0^1 r^5 \sin\varphi\cos\varphi\,(-t^3)\, dt \right] d\varphi \right] dr = \int_0^2 \left[\int_{\frac{\pi}{2}}^{\pi} r^5 \sin\varphi\cos\varphi \underbrace{\left[-\frac{t^4}{4} \right]_0^1}_{-\frac{1}{4}} d\varphi \right] dr =$$

$$= -\frac{1}{4} \int_0^2 \left[\int_{\frac{\pi}{2}}^{\pi} r^5 \cdot \frac{1}{2}\sin 2\varphi\, d\varphi \right] dr = -\frac{1}{8} \int_0^2 \left[\int_{\frac{\pi}{2}}^{\pi} r^5 \sin 2\varphi\, d\varphi \right] dr =$$

$$= -\frac{1}{8} \int_0^2 r^5 \cdot \left[-\frac{1}{2}\cos 2\varphi \right]_{\frac{\pi}{2}}^{\pi} dr = \frac{1}{16} \int_0^2 r^5 \left(\underbrace{\cos 2\pi}_{1} - \underbrace{\cos\pi}_{-1} \right) dr =$$

$$= \frac{1}{16} \cdot 2 \left[\frac{r^6}{6} \right]_0^2 = \frac{1}{8} \cdot \frac{2^6}{6} = \frac{2^6}{2^4 \cdot 3} = \frac{2^2}{3} = \underline{\underline{\frac{4}{3}}}$$