TRANSFORMACE TROJNÉHO INTEGRÁLU
DO CYLINDRICKÝCH SOUŘADNIC

Transformaci do cylindrických souřadnic využíváme, je-li integrační obor ohraničen:

- rotační válcovou plochou
- rotační kuželovou plochou
- rotačním eliptickým paraboloidem

osu těchto ploch označíme o

(1) $o \parallel z$:

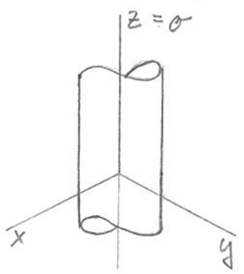
$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \text{ Polární souřadnice}$$
$$z = z$$

$J = r$ ... Jakobián

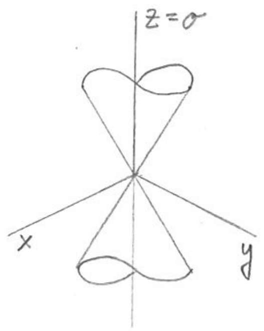
$\rightarrow x^2 + y^2 = r^2$

$$\rightarrow \iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W'} f(r\cos\varphi, r\sin\varphi, z)\cdot r\,dr\,d\varphi\,dz$$

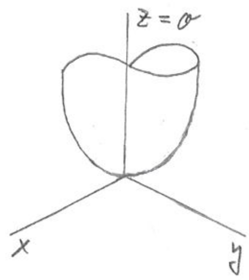
* rotační válcová plocha
např. $x^2+y^2=a^2$



rotační kuželová plocha
např. $z^2=x^2+y^2$



rotační eliptický paraboloid
např. $z=x^2+y^2$



Z transformačních rovnic je zřejmé, že proměnné x a y nahrazujeme proměnnými r a φ stejně jako u polárních souřadnic. $\Rightarrow$

$\Rightarrow$ Meze pro proměnné r a φ určujeme stejně jako u polárních souřadnic (z průmětu tělesa do roviny (x,y)).

Proměnná z se nemění; meze pro proměnnou z je ale třeba vyjádřit pomocí nových proměnných r a φ.

Obdobně postupujeme i u případů (2) a (3).

(2) $o \parallel y$: *

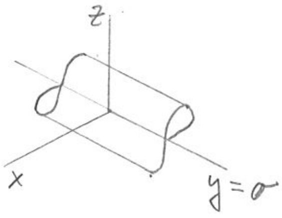
$$x = r\cos\varphi$$
$$y = y$$
$$z = r\sin\varphi$$

$$J = r$$

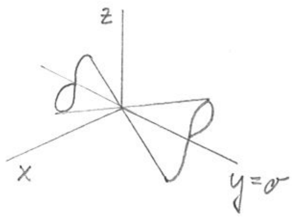
$\rightarrow x^2 + z^2 = r^2$

$\rightarrow \iiint\limits_{W} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{W'} f(r\cos\varphi,\ y,\ r\sin\varphi)\cdot r\,dr\,d\varphi\,dy$

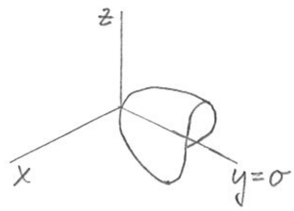
* rotační válcová plocha
např. $x^2 + z^2 = a^2$



rotační kuželová plocha
např. $y^2 = x^2 + z^2$



rotační eliptický paraboloid
např. $y = x^2 + z^2$



(3) $o \parallel x$: *

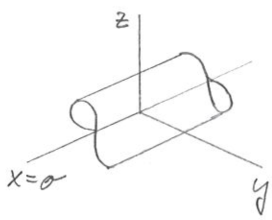
$$x = x$$
$$y = r\cos\varphi$$
$$z = r\sin\varphi$$

$$J = r$$

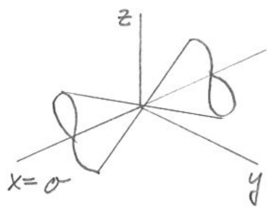
$\rightarrow y^2 + z^2 = r^2$

$\rightarrow \iiint\limits_{W} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{W'} f(x,\ r\cos\varphi,\ r\sin\varphi)\cdot r\,dr\,d\varphi\,dx$

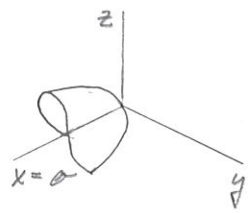
* rotační válcová plocha
např. $y^2 + z^2 = a^2$



rotační kuželová plocha
např. $x^2 = y^2 + z^2$



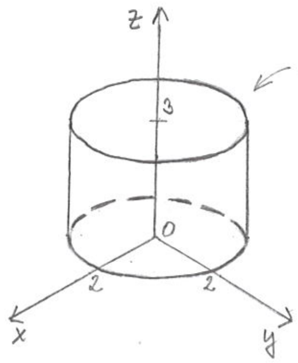
rotační eliptický paraboloid
např. $x = y^2 + z^2$



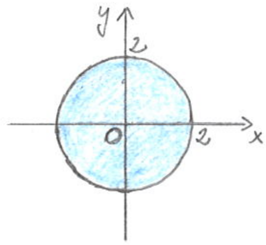
Př: Pomocí transformace do cylindrických souřadnic vypočtěte následující integrály.

1) $\iiint\limits_W y^2 z^3\, dx\,dy\,dz =$ $W: x^2+y^2 \leq 4,\ z \geq 0,\ z \leq 3$

$x^2+y^2=4$ ... rotační válcová plocha o poloměru 2, osa $o=z$



Průmět do (x,y):



$0 \leq \varphi \leq 2\pi$
$0 \leq r \leq 2$
$0 \leq z \leq 3$

$$= \int_0^{2\pi}\Big[\int_0^2\Big[\int_0^3 \underbrace{r^2\sin^2\varphi}_{y^2}\cdot z^3\cdot \underbrace{r}_{\text{jakobián}}\,dz\Big]dr\Big]d\varphi = \int_0^{2\pi}\Big[\int_0^2 r^3\sin^2\varphi\cdot\Big[\frac{z^4}{4}\Big]_0^3 dr\Big]d\varphi =$$

$$= \int_0^{2\pi}\Big[\int_0^2 r^3\sin^2\varphi\cdot\frac{81}{4}\,dr\Big]d\varphi = \frac{81}{4}\int_0^{2\pi}\sin^2\varphi\cdot\Big[\frac{r^4}{4}\Big]_0^2 d\varphi = \frac{81}{4}\int_0^{2\pi}\sin^2\varphi\cdot\frac{16}{4}\,d\varphi =$$

$$= 81\int_0^{2\pi}\frac{1-\cos 2\varphi}{2}\,d\varphi = \frac{81}{2}\Big[\varphi-\frac{1}{2}\sin 2\varphi\Big]_0^{2\pi} = \frac{81}{2}\Big(2\pi-\frac{1}{2}\underbrace{\sin 4\pi}_{0}-0+\frac{1}{2}\underbrace{\sin 0}_{0}\Big) =$$

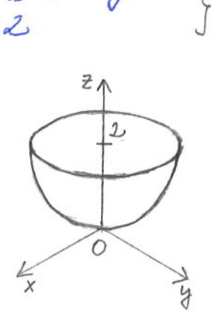
$$= \frac{81}{2}\cdot 2\pi = \underline{81\pi}$$

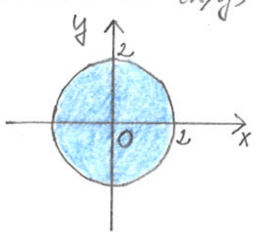
2) $\iiint\limits_W \underbrace{(x^2+y^2)}_{r^2}\,dx\,dy\,dz =$ $W: 2z \geq x^2+y^2,\ z \leq 2$

$2z = x^2+y^2 \Rightarrow z = \frac{1}{2}(x^2+y^2) = \frac{1}{2}r^2$ ... rotační eliptický paraboloid, osa $o=z$

$\left.\begin{array}{l} z=\frac{1}{2}(x^2+y^2)\\ z=2\end{array}\right\} \Rightarrow x^2+y^2=4$

Průmět do (x,y):





$0 \leq \varphi \leq 2\pi$
$0 \leq r \leq 2$
$\frac{1}{2}r^2 \leq z \leq 2$

$$= \int_0^{2\pi}\Big[\int_0^2\Big[\int_{\frac{1}{2}r^2}^2 r^2\cdot r\,dz\Big]dr\Big]d\varphi = \int_0^{2\pi}\Big[\int_0^2 r^3\cdot[z]_{\frac{1}{2}r^2}^2\,dr\Big]d\varphi = \int_0^{2\pi}\Big[\int_0^2 r^3\Big(2-\frac{1}{2}r^2\Big)dr\Big]d\varphi =$$

$$= \int_0^{2\pi}\Big[\int_0^2\Big(2r^3-\frac{1}{2}r^5\Big)dr\Big]d\varphi = \int_0^{2\pi}\Big[\frac{r^4}{2}-\frac{r^6}{12}\Big]_0^2 d\varphi = \int_0^{2\pi}\Big(8-\frac{16}{3}\Big)d\varphi = \frac{8}{3}[\varphi]_0^{2\pi} = \underline{\frac{16}{3}\pi}$$

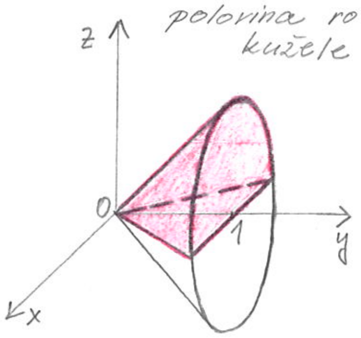
3) $\iiint\limits_W y\,dx\,dy\,dz =$ $\qquad$ $W:\ y \geq \sqrt{x^2+z^2},\ y \leq 1,\ z \geq 0$

$y = \sqrt{x^2+z^2} = \sqrt{r^2} = r$

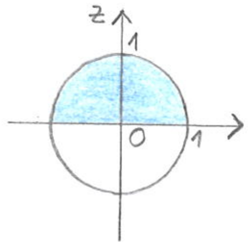
$y^2 = x^2+z^2$ ... rotační kuželová plocha, osa $o = y$ $\quad \Rightarrow \quad x = r\cos\varphi$, $y = y$, $z = r\sin\varphi$

$\left.\begin{array}{l} y^2 = x^2+z^2 \\ y = 1 \end{array}\right\} \Rightarrow x^2+z^2 = 1$

polovina rotačního kužele



průmět do (x,z)



$0 \leq \varphi \leq \pi$
$0 \leq r \leq 1$
$r \leq y \leq 1$

$$= \int_0^{\pi}\Big[\int_0^1\Big[\int_r^1 y\,r\,dy\Big]dr\Big]d\varphi = \int_0^{\pi}\Big[\int_0^1 r\Big[\frac{y^2}{2}\Big]_r^1 dr\Big]d\varphi = \frac{1}{2}\int_0^{\pi}\Big[\int_0^1 r(1-r^2)\,dr\Big]d\varphi =$$

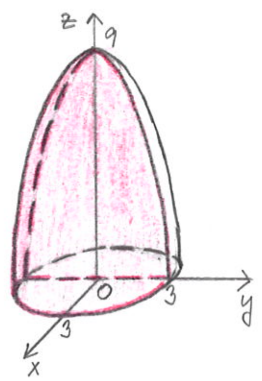
$$= \frac{1}{2}\int_0^{\pi}\Big[\int_0^1 (r-r^3)\,dr\Big]d\varphi = \frac{1}{2}\int_0^{\pi}\Big[\frac{r^2}{2}-\frac{r^4}{4}\Big]_0^1 d\varphi = \frac{1}{2}\int_0^{\pi}\Big(\frac{1}{2}-\frac{1}{4}\Big)d\varphi = \frac{1}{2}\cdot\frac{1}{4}\int_0^{\pi}d\varphi =$$

$$= \frac{1}{8}\big[\varphi\big]_0^{\pi} = \underline{\underline{\frac{1}{8}\pi}}$$

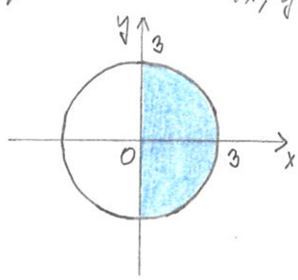
4) $\iiint\limits_W \frac{x}{(9-z)^2}\,dx\,dy\,dz =$ $\qquad$ $W:\ z \leq 9-x^2-y^2,\ z \geq 0,\ x \geq 0$

$\begin{cases} z = 9-x^2-y^2 = 9-(x^2+y^2) = 9-r^2 \\ z = 0 \end{cases}$ ... rotační eliptický paraboloid, osa $o = z$

$9-x^2-y^2 = 0 \Rightarrow x^2+y^2 = 9$



průmět do (x,y)



$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$
$0 \leq r \leq 3$
$0 \leq z \leq 9-r^2$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big[\int_0^3\Big[\int_0^{9-r^2} \frac{r\cos\varphi}{(9-z)^2}\cdot r\,dz\Big]dr\Big]d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big[\int_0^3\Big[\int_0^{9-r^2} r^2\cos\varphi\cdot\frac{1}{(9-z)^2}\,dz\Big]dr\Big]d\varphi \overset{*}{=}$$

$$* \int \frac{1}{(9-z)^2}\,dz = \left|\begin{array}{l} t = 9-z \\ dt = -dz \end{array}\right| = -\int \frac{1}{t^2}\,dt = -\int t^{-2}\,dt = -\frac{t^{-1}}{-1} = \frac{1}{t} = \frac{1}{9-z}$$

$$\overset{*}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^3 r^2\cos\varphi\cdot\left[\frac{1}{9-z}\right]_0^{9-r^2} dr\right]d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^3 r^2\cos\varphi\cdot\left(\frac{1}{r^2}-\frac{1}{9}\right)dr\right]d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^3 \cos\varphi\cdot\left(1-\frac{1}{9}r^2\right)dr\right]d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\cdot\left[r-\frac{1}{27}r^3\right]_0^3 dr =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\cdot(3-1)\,dr = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\,d\varphi = 2\left[\sin\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\Big(\underbrace{\sin\frac{\pi}{2}}_{1}-\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1}\Big) =$$

$$= 2(1+1) = \underline{4}$$

5) $\iiint_W z\underbrace{\sqrt{x^2+y^2}}_{\sqrt{r^2}=r}\,dx\,dy\,dz =$

$W:\ y \geq 0,\ z \geq 0,\ z \leq 3,\ x^2+y^2 \leq 2x$ *

$\rightarrow (x-1)^2+y^2=1$

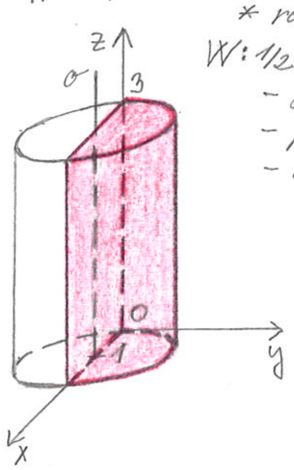
$\rightarrow r^2 = 2r\cos\varphi$

$r = 2\cos\varphi$

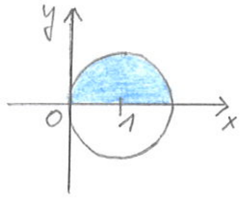
* rotační válcová plocha

W: 1/2 rotačního válce:

- $o \parallel z$
- poloměr: $a=1$
- výška: $v=3$



Průmět do (x,y):



$0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq r \leq 2\cos\varphi$

$0 \leq z \leq 3$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^{2\cos\varphi}\left[\int_0^3 z\cdot r\cdot r\,dz\right]dr\right]d\varphi = \int_0^{\frac{\pi}{2}}\left[\int_0^{2\cos\varphi} r^2\cdot\left[\frac{z^2}{2}\right]_0^3 dr\right]d\varphi = \int_0^{\frac{\pi}{2}}\left[\int_0^{2\cos\varphi}\frac{9}{2}r^2\,dr\right]d\varphi =$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^{2\cos\varphi}d\varphi = \frac{9}{2}\cdot\frac{1}{3}\int_0^{\frac{\pi}{2}} 8\underbrace{\cos^3\varphi}_{\cos^2\varphi\,\cdot\,\cos\varphi}\,d\varphi = 12\int_0^{\frac{\pi}{2}}\underbrace{(1-\sin^2\varphi)}_{\cos^2\varphi}\cdot\cos\varphi\,d\varphi =$$

$$= \left|\begin{matrix} t=\sin\varphi \\ dt=\cos\varphi\,d\varphi \end{matrix}\quad \begin{array}{c|c|c} \varphi & 0 & \frac{\pi}{2} \\ \hline t & 0 & 1 \end{array}\right| = 12\int_0^1(1-t^2)\,dt = 12\left[t-\frac{t^3}{3}\right]_0^1 = 12\left(1-\frac{1}{3}\right) =$$

$$= 12\cdot\frac{2}{3} = \underline{8}$$