TROJNÝ INTEGRÁL
NA OBLASTECH PRVNÍHO, DRUHÉHO A TŘETÍHO DRUHU

Elementární oblasti v prostoru:

- Oblast I. druhu:

  $[x,y] \in A$ ... oblast I. nebo II. druhu v rovině (x,y)

  $g_1(x,y) \leq z \leq h_1(x,y)$

  - nejprve integrujeme podle proměnné z

- Oblast II. druhu:

  $[x,z] \in B$ ... oblast I. nebo II. druhu v rovině (x,z)

  $g_2(x,z) \leq y \leq h_2(x,z)$

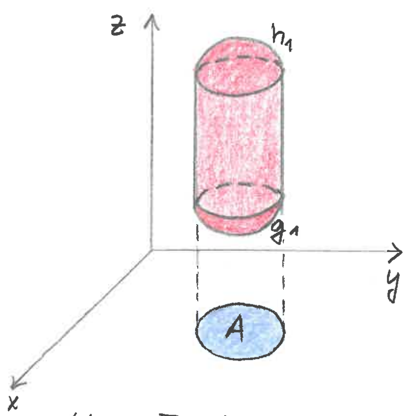
  - nejprve integrujeme podle proměnné y

- Oblast III. druhu:

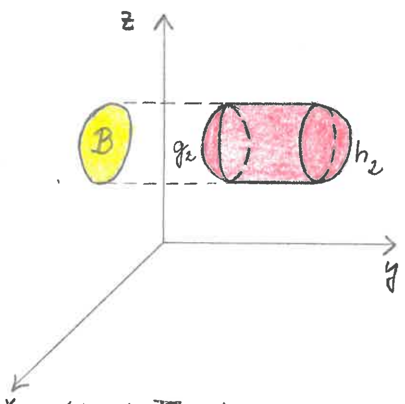
  $[y,z] \in C$ ... oblast I. nebo II. druhu v rovině (y,z)

  $g_3(y,z) \leq x \leq h_3(y,z)$

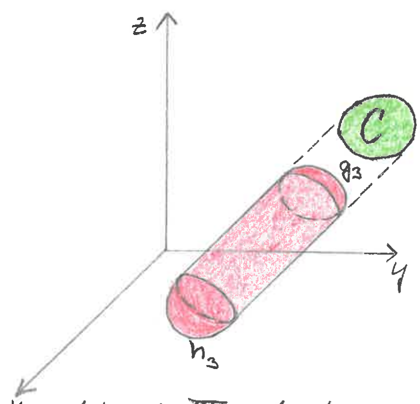
  - nejprve integrujeme podle proměnné x



oblast I. druhu



oblast II. druhu



oblast III. druhu

V příkladech se setkáme téměř výhradně jen s trojným integrálem na oblasti I. druhu, tj.

$$\left.\begin{array}{l} a \le x \le b \\ \alpha(x) \le y \le \beta(x) \end{array}\right\} \text{oblast I. druhu v rovině } (x,y)$$
$$g(x,y) \le z \le h(x,y)$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \int_a^b \left[\int_{\alpha(x)}^{\beta(x)} \left[\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz \right] dy \right] dx$$

nebo

$$\left.\begin{array}{l} c \le y \le d \\ \gamma(y) \le x \le \delta(y) \end{array}\right\} \text{oblast II. druhu v rovině } (x,y)$$
$$g(x,y) \le z \le h(x,y)$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \int_c^d \left[\int_{\gamma(y)}^{\delta(y)} \left[\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz \right] dx \right] dy$$

- vnější meze: „od čísla k číslu“ – pevné
- střední meze: „od křivky k křivce“ – fce jedné proměnné
- vnitřní meze: „od plochy k ploše“ – fce dvou proměnných

} *

* i při integraci na oblastech II. a III. druhu (tj. při jiném pořadí integrace)

Př: Vypočtěte integrály na množině W.

1) $$\iiint_W \frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz =$$

W: $x \geq 0,\ y \geq 0,\ z \geq 0,\ \overbrace{x+y+z \leq 1}^{*}$

$\rightarrow$ $x=0$ ... rovina (y,z)

$y=0$ ... rovina (x,z)

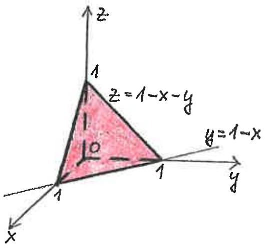
$z=0$ ... rovina (x,y)

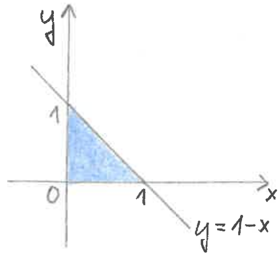
$*$ $x+y+z=1$ ... rovina

$\rightarrow$ $z = 1-x-y$ ... horní mez pro z

$\rightarrow$ $z=0$: $x+y=1$ ... průsečnice roviny $x+y+z=1$ s rovinou (x,y)

$y = 1-x$





$$\left.\begin{aligned} 0 &\leq x \leq 1 \\ 0 &\leq y \leq 1-x \\ 0 &\leq z \leq 1-x-y \end{aligned}\right\} \quad (*)$$

Množina W je oblast I., II. i III. druhu. Lze tedy zvolit libovolné pořadí integrace.
„Nejpřirozenější" volba pořadí integrace je ale:
1. podle z, 2. podle y, 3. podle x.
Při tomto pořadí jsou meze dány nerovnostmi $(*)$.

$$= \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz\right] dy\right] dx \overset{⊛}{=} \int_0^1 \left[\int_0^{1-x} \left[-\frac{1}{2(x+y+z+1)^2}\right]_0^{1-x-y} dy\right] dx = -\frac{1}{2}\int_0^1 \left[\int_0^{1-x} \left(\frac{1}{4} - \frac{1}{(x+y+1)^2}\right) dy\right] dx =$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x} dx = -\frac{1}{2}\int_0^1 \left(\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{x+1}\right) dx = -\frac{1}{2}\int_0^1 \left(\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right) dx =$$

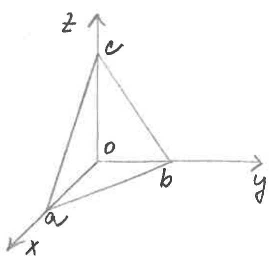
$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{1}{8}x^2 - \ln|x+1|\right]_0^1 = -\frac{1}{2}\left(\frac{3}{4} - \frac{1}{8} - \ln 2\right) = \underline{\underline{\frac{1}{2}\left(\ln 2 - \frac{5}{8}\right)}}$$

⊛ $$\int \frac{1}{(x+y+z+1)^3}\,dz = \left| \begin{aligned} t &= x+y+z+1 \\ dt &= dz \end{aligned} \right| = \int \frac{1}{t^3}\,dt = \int t^{-3}\,dt = \frac{t^{-2}}{-2} = -\frac{1}{2t^2} = -\frac{1}{2(x+y+z+1)^2}$$

Obdobně vypočteme $\displaystyle\int \frac{1}{(x+y+1)^2}\,dy$

Pozn: Úsekový tvar roviny:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

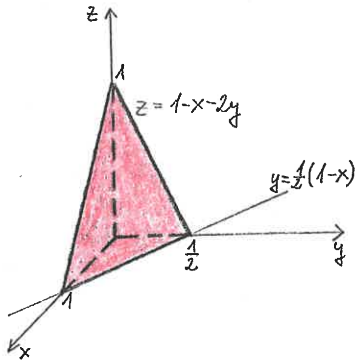


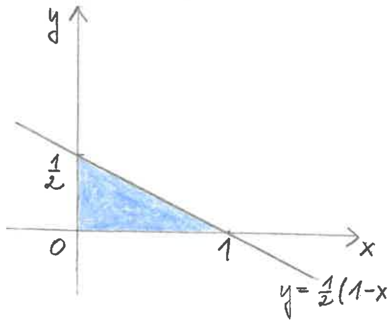
2) $\iiint\limits_W \frac{1}{1-x}\,dx\,dy\,dz =$ W: $\underbrace{x+2y+z=1}_{*}$, $x=0$, $y=0$, $z=0$

* • $\frac{x}{1}+\frac{y}{\frac{1}{2}}+\frac{z}{1}=1$ ... úsekový tvar roviny

• $z = 1-x-2y$

• $z=0$: $x+2y=1$
$y=\frac{1}{2}(1-x)$





$0 \le x \le 1$
$0 \le y \le \frac{1}{2}(1-x)$
$0 \le z \le 1-x-2y$

$$= \int_0^1 \left[\int_0^{\frac{1}{2}(1-x)} \left[\int_0^{1-x-2y} \frac{1}{1-x}\,dz \right] dy \right] dx = \int_0^1 \left[\int_0^{\frac{1}{2}(1-x)} \frac{1}{1-x}\cdot [z]_0^{1-x-2y}\,dy \right] dx = \int_0^1 \left[\int_0^{\frac{1}{2}(1-x)} \frac{1-x-2y}{1-x}\,dy \right] dx =$$

$$= \int_0^1 \left[\int_0^{\frac{1}{2}(1-x)} \left(1-\frac{2y}{1-x}\right) dy \right] dx = \int_0^1 \left[y - \frac{y^2}{1-x} \right]_0^{\frac{1}{2}(1-x)} dx = \int_0^1 \left(\frac{1}{2}(1-x) - \frac{\frac{1}{4}(1-x)^2}{1-x} \right) dx =$$

$$= \int_0^1 \left(\frac{1}{2}(1-x) - \frac{1}{4}(1-x) \right) dx = \frac{1}{4}\int_0^1 (1-x)\,dx = \frac{1}{4}\left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{4}\cdot\frac{1}{2} = \underline{\underline{\frac{1}{8}}}$$

3) $\iiint\limits_W e^y\,dx\,dy\,dz =$ W: $0 \le y \le 3$, $\underbrace{x^2 \le z \le 4}_{*}$

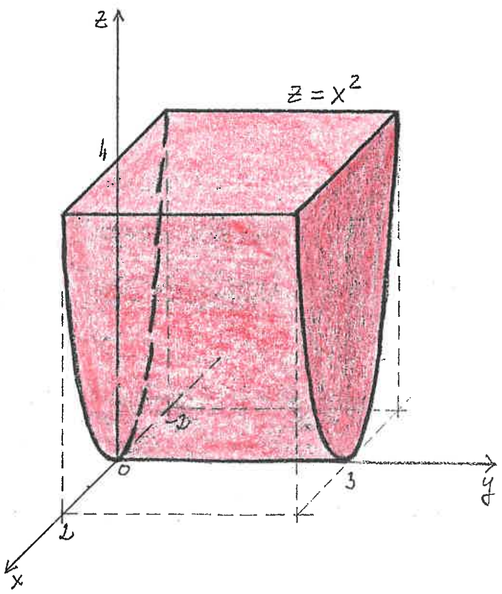
* $z = x^2$ ... parabolická válcová plocha
$z = 4$ ... rovina rovnoběžná s (x,y) $\Big\} \Rightarrow$ $x^2 = 4$
$x = \pm 2$ ... meze pro x

$$= \int_{-2}^{2} \left[\int_0^3 \left[\int_{x^2}^{4} e^y\,dz \right] dy \right] dx = \int_{-2}^{2} \left[\int_0^3 e^y [z]_{x^2}^{4}\,dy \right] dx =$$

$$= \int_{-2}^{2} \left[\int_0^3 (4-x^2)e^y\,dy \right] dx = \int_{-2}^{2} (4-x^2)[e^y]_0^3\,dx =$$

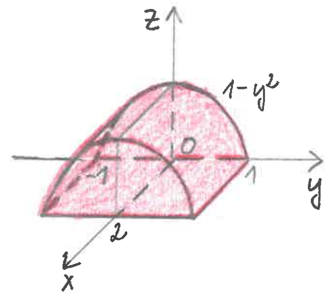
$$= \int_{-2}^{2} (4-x^2)(e^3-1)\,dx = (e^3-1)\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} =$$

$$= (e^3-1)\left(8-\frac{8}{3}+8-\frac{8}{3}\right) = \underline{\underline{\frac{32}{3}(e^3-1)}}$$



4) $\iiint\limits_W \frac{x}{y+1}\,dx\,dy\,dz =$ $W:\ x=0,\ x=2,\ z=0,\ z=1-y^2$ parabolická válcová plocha

meze pro y: $\left.\begin{array}{l} z=0 \\ z=1-y^2 \end{array}\right\} \Rightarrow \begin{array}{l} 1-y^2=0 \\ y^2=1 \\ y=\pm 1 \end{array}$



$-1 \le y \le 1$
$0 \le x \le 2$
$0 \le z \le 1-y^2$

$$= \int_{-1}^{1}\Big[\int_0^2\Big[\int_0^{1-y^2} \frac{x}{y+1}\,dz\Big]dx\Big]dy =$$

$$= \int_{-1}^{1}\Big[\int_0^2 \frac{x}{y+1}\cdot[z]_0^{1-y^2}dx\Big]dy = \int_{-1}^{1}\Big[\int_0^2 \frac{x}{y+1}\cdot(1-y^2)\,dx\Big]dy = \int_{-1}^{1}\frac{1-y^2}{1+y}\cdot\Big[\frac{x^2}{2}\Big]_0^2 dy =$$

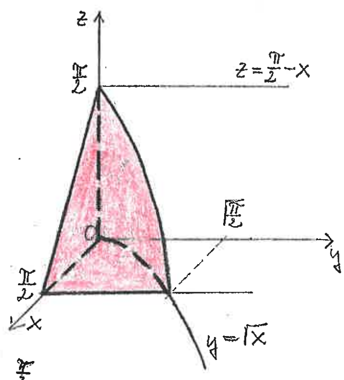
$$= \int_{-1}^{1}\frac{(1+y)(1-y)}{1+y}\cdot 2\,dy = 2\int_{-1}^{1}(1-y)\,dy = 2\Big[y-\frac{y^2}{2}\Big]_{-1}^{1} = 2\Big(1-\frac{1}{2}+1+\frac{1}{2}\Big) = \underline{\underline{4}}$$

5) $\iiint\limits_W y\cos(x+z)\,dx\,dy\,dz =$ $W:\ y=0,\ z=0,\ y=\sqrt{x},\ x+z=\frac{\pi}{2}$

- $y=\sqrt{x} \ \Rightarrow\ x=y^2$ … parabolická válcová plocha
- $x+z=\frac{\pi}{2}$ … rovina rovnoběžná s osou y (kolmá k (x,z))
  - $\to z=\frac{\pi}{2}-x$
  - $\to z=0:\ x=\frac{\pi}{2}$ … průsečnice s (x,y)

$$= \int_0^{\frac{\pi}{2}}\Big[\int_0^{\sqrt{x}}\Big[\int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,dz\Big]dy\Big]dx = \int_0^{\frac{\pi}{2}}\Big[\int_0^{\sqrt{x}} y\,[\sin(x+z)]_0^{\frac{\pi}{2}-x}\,dy\Big]dx =$$

$$= \int_0^{\frac{\pi}{2}}\Big[\int_0^{\sqrt{x}} y\Big(\sin\frac{\pi}{2}-\sin x\Big)dy\Big]dx = \int_0^{\frac{\pi}{2}}\Big[\int_0^{\sqrt{x}} y(1-\sin x)\,dy\Big]dx =$$



$$= \int_0^{\frac{\pi}{2}}(1-\sin x)\Big[\frac{y^2}{2}\Big]_0^{\sqrt{x}}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} x(1-\sin x)\,dx =$$

$$= \frac{1}{2}\Big(\int_0^{\frac{\pi}{2}} x\,dx - \int_0^{\frac{\pi}{2}} x\sin x\,dx\Big) = \left|\begin{array}{ll} u=x & v'=\sin x \\ u'=1 & v=-\cos x \end{array}\right| = \frac{1}{2}\Big(\Big[\frac{x^2}{2}\Big]_0^{\frac{\pi}{2}} + [x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\cos x\,dx\Big) =$$

$$= \frac{1}{2}\Big(\frac{1}{2}[x^2]_0^{\frac{\pi}{2}} + [x\cos x]_0^{\frac{\pi}{2}} - [\sin x]_0^{\frac{\pi}{2}}\Big) = \frac{1}{2}\Big(\frac{\pi^2}{8}+0-1\Big) = \underline{\underline{\frac{1}{16}(\pi^2-8)}}$$