TRANSFORMACE DVOJNÉHO INTEGRÁLU
DO POLÁRNÍCH SOUŘADNIC

Obecná transformace:

$$\left.\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}\right\} \text{ transformační rovnice}$$

$$\rightarrow \iint_D f(x,y)\,dx\,dy = \iint_{D'} f(g(u,v),\, h(u,v)) \cdot |J|\,du\,dv$$

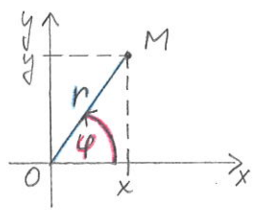
kde J se nazývá jakobián a

$$J = \begin{vmatrix} g'_u & g'_v \\ h'_u & h'_v \end{vmatrix}$$

Účelem transformace je zjednodušit integrační obor nebo integrovanou funkci.

Polární souřadnice

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$



$M[x,y] \rightarrow M[r,\varphi]$

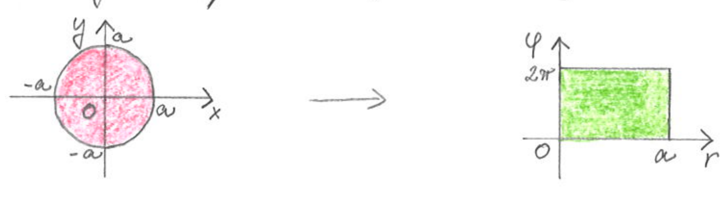
$$J = \begin{vmatrix} (r\cos\varphi)'_r & (r\cos\varphi)'_\varphi \\ (r\sin\varphi)'_r & (r\sin\varphi)'_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r\,\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1} = r$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(r\cos\varphi,\, r\sin\varphi)\cdot r\,dr\,d\varphi$$

Protože
$x^2 + y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2(\cos^2\varphi + \sin^2\varphi) = r^2$
budeme výraz x^2+y^2 vždy nahrazovat výrazem r^2.

Transformaci do polárních souřadnic využijeme, je-li integrační obor ohraničen kružnicí.

$D:\ x^2+y^2 \le a^2, a>0 \longrightarrow D':\ \langle 0, a\rangle \times \langle 0, 2\pi\rangle$



Pomocí transformace do polárních souřadnic jsme kruh převedli na obdélník (dvojrozměrný interval).

$$-a \le x \le a \qquad \longrightarrow \qquad 0 \le r \le a$$
$$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2} \qquad\qquad 0 \le \varphi \le 2\pi$$

Pro správné určení mezí pro proměnné r a φ je třeba množinu D zakreslit.

Vždy musí platit:

1. Rozsah mezí pro proměnnou φ musí být maximálně 2π.
2. Meze pro proměnnou r nesmí být záporné (proměnná r určuje vzdálenost od počátku!).

Pozn.:

1) Množina D je ohraničena kružnicí se středem v počátku, tj. kružnicí $x^2+y^2=a^2$, $a>0$, pak jsou meze pro proměnné r i φ pevné a nezáleží na pořadí integrace.

2) Množina D je ohraničena kružnicí

$x^2+y^2-ax=0$, $a>0$ nebo $x^2+y^2-ay=0$, $a>0$

$\to (x-\frac{a}{2})^2+y^2=\frac{a^2}{4}$

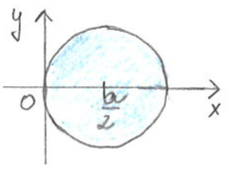
$\to r^2-ar\cos\varphi=0$

$\to r=a\cos\varphi$ ⊛

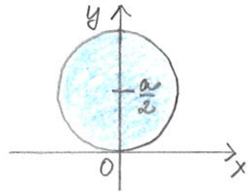
$\to x^2+(y-\frac{a}{2})^2=\frac{a^2}{4}$

$\to r^2-ar\sin\varphi=0$

$\to r=a\sin\varphi$ ⊛



$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$
$0\le r\le a\cos\varphi$



$0\le\varphi\le\pi$
$0\le r\le a\sin\varphi$

Pevné meze jsou jen pro proměnnou φ $\Rightarrow$
$\Rightarrow$ Pořadí integrace: 1. podle proměnné r
2. podle proměnné φ

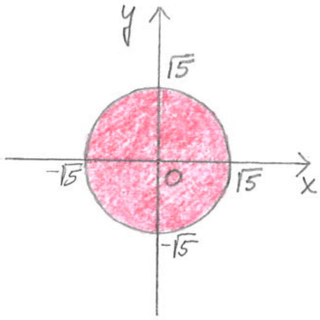
⊛ Rovnice dané kružnice v polárních souřadnicích.

Př.: Vypočtěte integrály s použitím transformace do polárních souřadnic.

1) $\iint\limits_D (1-2x-3y)\,dx\,dy =$ $\qquad D:\ x^2+y^2 \le 5$

$$= \int_0^{\sqrt5} \Big[\int_0^{2\pi} (1-2\underbrace{r\cos\varphi}_{x} - 3\underbrace{r\sin\varphi}_{y})\overset{\text{jakobián}}{r}\, d\varphi \Big] dr =$$

$$\boxed{\begin{aligned} 0 \le r \le \sqrt5 \\ 0 \le \varphi \le 2\pi \end{aligned}}$$



$$= \int_0^{\sqrt5} r\,[\varphi - 2r\sin\varphi + 3r\cos\varphi]_0^{2\pi}\, dr =$$

$$= \int_0^{\sqrt5} r\,(2\pi - 2r\underbrace{\sin 2\pi}_{0} + 3r\underbrace{\cos 2\pi}_{1} - 0 + 2r\underbrace{\sin 0}_{0} - 3r\underbrace{\cos 0}_{1})\, dr =$$

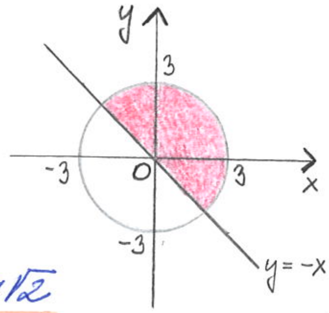
$$= \int_0^{\sqrt5} r\,(2\pi + 3r - 3r)\,dr = 2\pi \int_0^{\sqrt5} r\,dr = 2\pi \left[\frac{r^2}{2}\right]_0^{\sqrt5} = 2\pi \cdot \frac{(\sqrt5)^2}{2} = \underline{\underline{5\pi}}$$

2) $\iint\limits_D y\,dx\,dy =$ $\qquad D:\ x^2+y^2 \le 9\ ,\ \ y \ge -x$

$$r \cdot r\sin\varphi = y \quad (r \to \text{jakobián})$$

$$= \int_{-\frac{\pi}{4}}^{\frac34\pi} \Big[\int_0^3 r^2 \sin\varphi\, dr \Big] d\varphi =$$

$$\boxed{\begin{aligned} 0 \le r \le 3 \\ -\tfrac{\pi}{4} \le \varphi \le \tfrac34\pi \end{aligned}}$$



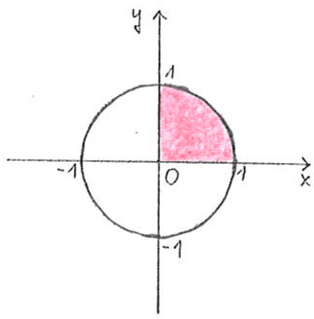
$$= \int_{-\frac{\pi}{4}}^{\frac34\pi} \left[\frac{r^3}{3}\right]_0^3 \sin\varphi\, d\varphi = 9\int_{-\frac{\pi}{4}}^{\frac34\pi} \sin\varphi\, d\varphi =$$

$$= 9\,[-\cos\varphi]_{-\frac{\pi}{4}}^{\frac34\pi} = 9\,(\underbrace{-\cos\tfrac34\pi}_{-\frac{\sqrt2}{2}} + \underbrace{\cos(-\tfrac{\pi}{4})}_{\frac{\sqrt2}{2}}) = 9\left(\frac{\sqrt2}{2} + \frac{\sqrt2}{2}\right) = \underline{\underline{9\sqrt2}}$$

3) $\iint\limits_D \underbrace{\sqrt{1-x^2-y^2}}_{1-(x^2+y^2)\,=\,1-r^2}\,dx\,dy =$ $\qquad D:\ x^2+y^2 \le 1,\ \ x \ge 0,\ \ y \ge 0$

$$= \int_0^1 \Big[\int_0^{\frac{\pi}{2}} r\sqrt{1-r^2}\, d\varphi \Big] dr = \int_0^1 r\sqrt{1-r^2}\,[\varphi]_0^{\frac{\pi}{2}}\, dr =$$

$$\boxed{\begin{aligned} 0 \le r \le 1 \\ 0 \le \varphi \le \tfrac{\pi}{2} \end{aligned}}$$



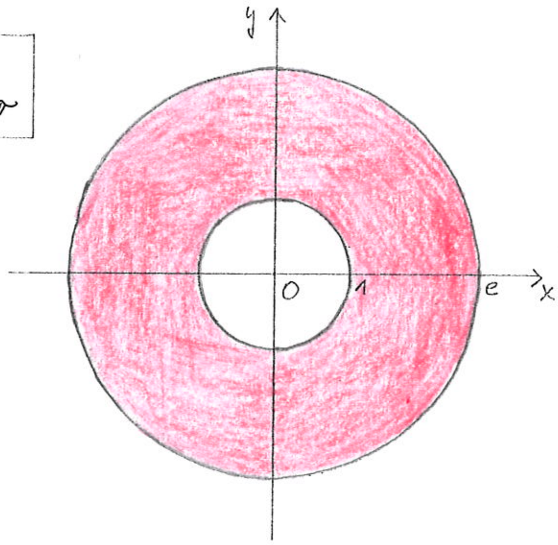
$$= \frac{\pi}{2}\int_0^1 r\sqrt{1-r^2}\, dr = \left| \begin{array}{l} t = 1-r^2 \\ dt = -2r\,dr \end{array} \quad \begin{array}{c|c|c} r & 0 & 1 \\ \hline t & 1 & 0 \end{array} \right| =$$

$$= \frac{\pi}{4}\int_0^1 \sqrt{t}\, dt = \frac{\pi}{4}\left[\frac23\sqrt{t^3}\right]_0^1 = \underline{\frac{\pi}{6}}$$

4) $\iint\limits_D \frac{\ln(x^2+y^2)}{x^2+y^2}\,dx\,dy =$

D: $1 \le x^2+y^2 \le e^2$

$$\boxed{\begin{matrix} 1 \le r \le e \\ 0 \le \varphi \le 2\pi \end{matrix}}$$



$$= \int_1^e \left[\int_0^{2\pi} \frac{\ln r^2}{r^2} \cdot r\,d\varphi \right] dr = \int_1^e \left[\int_0^{2\pi} \frac{2\ln r}{r}\,d\varphi \right] dr =$$

$$= \int_1^e \frac{2\ln r}{r} \left[\varphi \right]_0^{2\pi} dr = 4\pi \int_1^e \frac{\ln r}{r}\,dr =$$

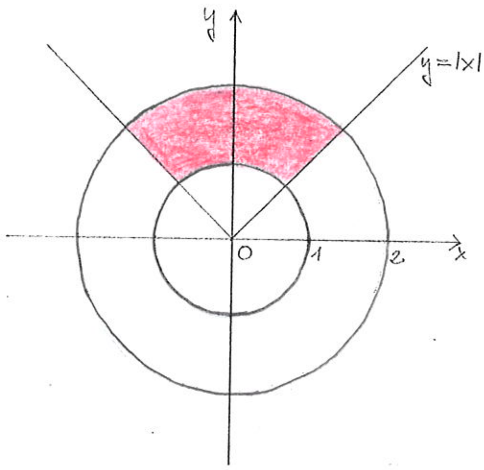
$$= \left| \begin{matrix} t = \ln r \\ dt = \frac{1}{r}dr \end{matrix} \quad \begin{array}{c|c|c} r & 1 & e \\ \hline t & 0 & 1 \end{array} \right| = 4\pi \int_0^1 t\,dt =$$

$$= 2\pi \left[t^2 \right]_0^1 = \underline{\underline{2\pi}}$$

5) $\iint\limits_D (x^2+y^2)\,dx\,dy =$

D: $1 \le x^2+y^2 \le 4$, $y \ge |x|$

$$\boxed{\begin{matrix} 1 \le r \le 2 \\ \frac{\pi}{4} \le \varphi \le \frac{3}{4}\pi \end{matrix}}$$



$$= \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \left[\int_1^2 r^3\,dr \right] d\varphi = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \left[\tfrac{1}{4} r^4 \right]_1^2 d\varphi =$$

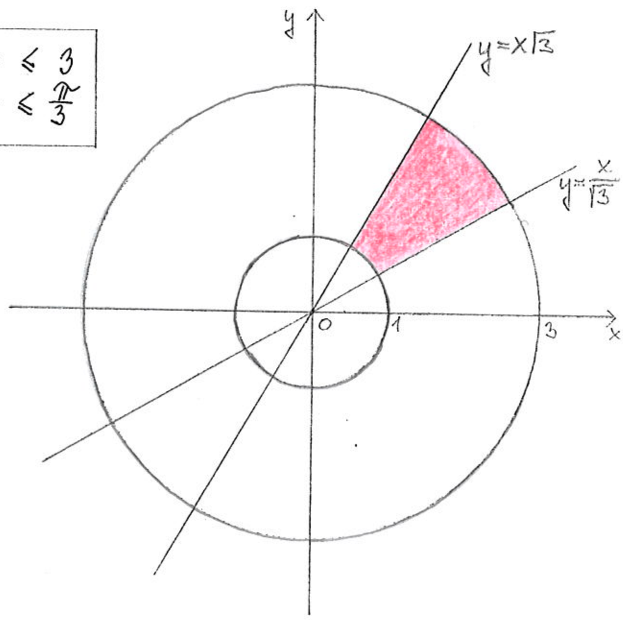
$$= \left(4 - \tfrac{1}{4}\right) \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi = \frac{15}{4} \left[\varphi \right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} =$$

$$= \frac{15}{4}\left(\tfrac{3}{4}\pi - \tfrac{\pi}{4} \right) = \frac{15}{4} \cdot \frac{\pi}{2} = \underline{\underline{\frac{15}{8}\pi}}$$

6) $\iint\limits_D \operatorname{arctg} \frac{y}{x}\,dx\,dy =$

D: $x^2+y^2 \ge 1$, $x^2+y^2 \le 9$, $y \ge \frac{x}{\sqrt{3}}$, $y \le x\sqrt{3}$

$$\boxed{\begin{matrix} 1 \le r \le 3 \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{3} \end{matrix}}$$



$$= \int_1^3 \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \cdot \operatorname{arctg} \overset{\operatorname{tg}\varphi}{\frac{r\sin\varphi}{r\cos\varphi}}\,d\varphi \right] dr =$$

$$= \int_1^3 \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \cdot \overbrace{\operatorname{arctg}(\operatorname{tg}\varphi)}^{\varphi}\,d\varphi \right] dr = \int_1^3 \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\varphi\,d\varphi \right] dr =$$

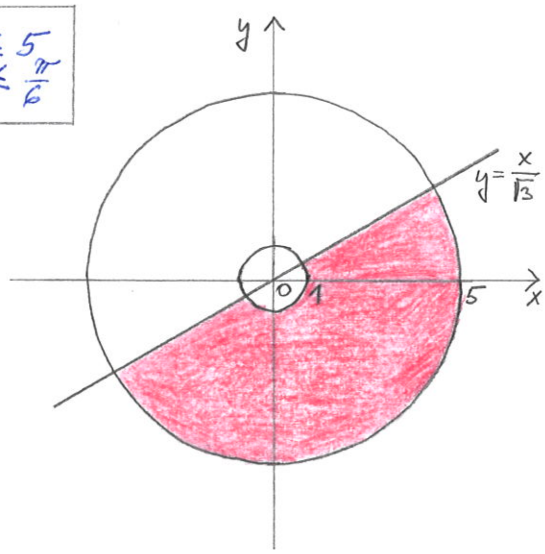
$$= \int_1^3 r \left[\tfrac{1}{2}\varphi^2 \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} dr = \tfrac{1}{2}\left(\frac{\pi^2}{9} - \frac{\pi^2}{36} \right) \int_1^3 r\,dr = \tfrac{1}{2} \cdot \frac{3\pi^2}{36} \int_1^3 r\,dr =$$

$$= \frac{\pi^2}{24} \left[\tfrac{1}{2} r^2 \right]_1^3 = \frac{\pi^2}{48} \cdot 8 = \underline{\underline{\frac{\pi^2}{6}}}$$

7) $\iint\limits_D \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy =$

$D:\ 1 \le x^2+y^2 \le 25,\quad y \le \frac{x}{\sqrt{3}}$

$$\boxed{\begin{matrix} 1 \le r \le 5 \\ -\frac{5}{6}\pi \le \varphi \le \frac{\pi}{6} \end{matrix}}$$



$$= \int_1^5 \left[\int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}} \frac{r\cos\varphi}{\sqrt{r^2}} \cdot r\,d\varphi \right] dr =$$

$$= \int_1^5 \left[\int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}} r\cos\varphi\,d\varphi \right] dr = \int_1^5 r\left[\sin\varphi\right]_{-\frac{5}{6}\pi}^{\frac{\pi}{6}} dr =$$

$$= \int_1^5 r\,(\underbrace{\sin\tfrac{\pi}{6}}_{\frac{1}{2}} - \underbrace{\sin(-\tfrac{5}{6}\pi)}_{-\frac{1}{2}})\,dr =$$

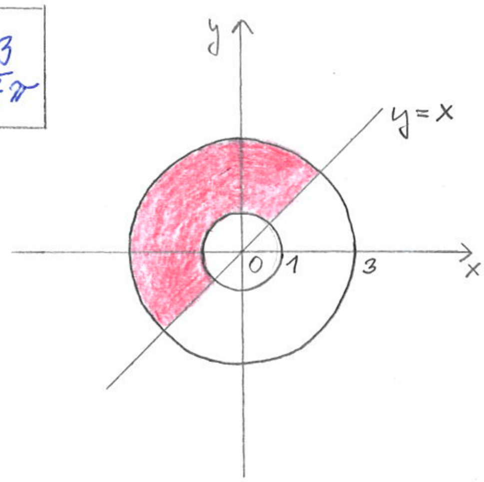
$$= \int_1^5 r\,(\tfrac{1}{2}+\tfrac{1}{2})\,dr = \int_1^5 r\,dr = \left[\frac{r^2}{2}\right]_1^5 =$$

$$= \frac{25}{2} - \frac{1}{2} = \frac{24}{2} = \underline{\underline{12}}$$

8) $\iint\limits_D e^{-x^2-y^2}\,dx\,dy =$

$D:\ 1 \le x^2+y^2 \le 9,\quad y \ge x$

$$\boxed{\begin{matrix} 1 \le r \le 3 \\ \frac{\pi}{4} \le \varphi \le \frac{5}{4}\pi \end{matrix}}$$



$$= \int_1^3 \left[\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} r\cdot e^{-r^2}\,d\varphi \right] dr = \int_1^3 r e^{-r^2} \left[\varphi\right]_{\frac{\pi}{4}}^{\frac{5}{4}\pi} dr =$$

$$= \int_1^3 r e^{-r^2} (\tfrac{5}{4}\pi - \tfrac{\pi}{4})\,dr = \pi \int_1^3 r e^{-r^2}\,dr =$$

$$= \left| \begin{matrix} t = -r^2 \\ dt = -2r\,dr \\ -\frac{1}{2}dt = r\,dr \end{matrix} \qquad \begin{array}{c|c|c} r & 1 & 3 \\ \hline t & -1 & -9 \end{array} \right| =$$

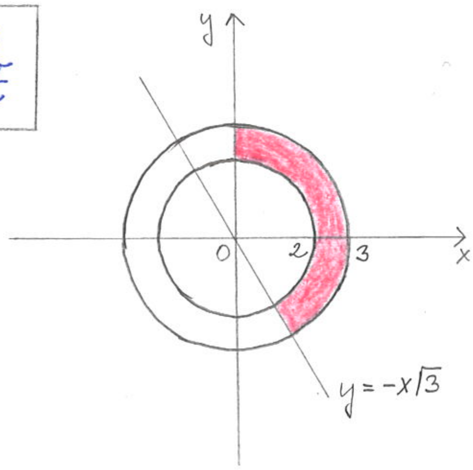
$$= -\frac{1}{2}\pi \int_{-1}^{-9} e^t\,dt = \frac{\pi}{2}\int_{-9}^{-1} e^t\,dt =$$

$$= \frac{\pi}{2}\left[e^t\right]_{-9}^{-1} = \underline{\underline{\frac{\pi}{2}(e^{-1}-e^{-9})}}$$

9) $\iint\limits_D \frac{y}{x^2+y^2}\,dx\,dy =$

$D:\ 4 \le x^2+y^2 \le 9,\quad y \ge -x\sqrt{3},\quad x \ge 0$

$$\boxed{\begin{matrix} 2 \le r \le 3 \\ -\frac{\pi}{3} \le \varphi \le \frac{\pi}{2} \end{matrix}}$$



$$= \int_2^3 \left[\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{r\sin\varphi}{r^2} \cdot r\,d\varphi \right] dr =$$

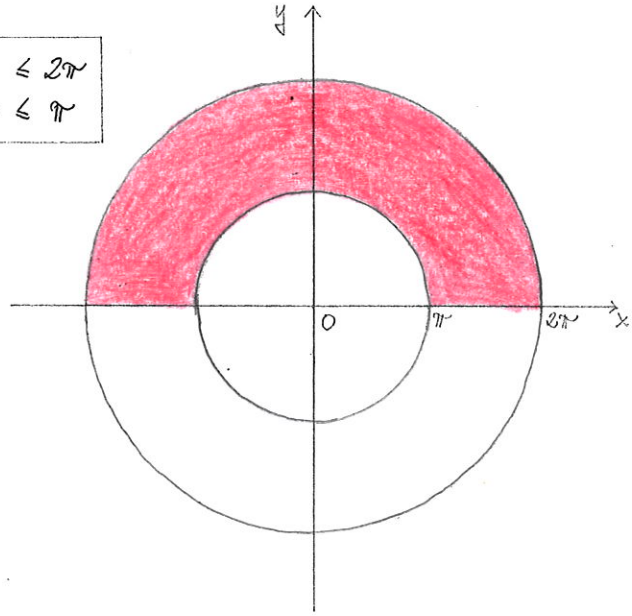
$$= \int_2^3 \left[\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\,d\varphi \right] dr = \int_2^3 \left[-\cos\varphi\right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}} dr =$$

$$= \int_2^3 (-\overbrace{\cos\tfrac{\pi}{2}}^{0} + \overbrace{\cos(-\tfrac{\pi}{3})}^{\frac{1}{2}})\,dr = \frac{1}{2}\int_2^3 dr =$$

$$= \frac{1}{2}\left[r\right]_2^3 = \frac{1}{2}(3-2) = \underline{\underline{\frac{1}{2}}}$$

10) $\iint_D \sin\sqrt{x^2+y^2}\,dx\,dy$ D: $\pi^2 \le x^2+y^2 \le 4\pi^2,\quad y \ge 0$

$$\boxed{\begin{matrix}\pi \le r \le 2\pi \\ 0 \le \varphi \le \pi\end{matrix}}$$



$$= \int_\pi^{2\pi} \left[\int_0^\pi r \sin r \, d\varphi \right] dr =$$

$$= \int_\pi^{2\pi} r \sin r \left[\varphi \right]_0^\pi dr = \pi \int_\pi^{2\pi} r \sin r \, dr =$$

$$= \left| \begin{matrix} u = r & v' = \sin r \\ u' = 1 & v = -\cos r \end{matrix} \right| =$$

$$= \pi \left(\left[-r \cos r \right]_\pi^{2\pi} + \int_\pi^{2\pi} \cos r \, dr \right) =$$

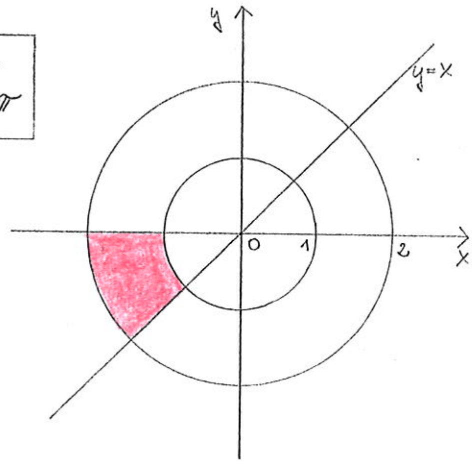
$$= \pi \left(\left[-r \cos r \right]_\pi^{2\pi} + \left[\sin r \right]_\pi^{2\pi} \right) =$$

$$= \pi \left(-2\pi \underbrace{\cos 2\pi}_{1} + \pi \underbrace{\cos \pi}_{-1} + \underbrace{\sin 2\pi}_{0} - \underbrace{\sin \pi}_{0} \right) =$$

$$= \pi (-2\pi - \pi) = \underline{-3\pi^2}$$

11) $\iint_D |x| y^2 \, dx\,dy$ D: $1 \le x^2+y^2 \le 4,\quad x \le y \le 0$

$$\boxed{\begin{matrix} 1 \le r \le 2 \\ \pi \le \varphi \le \frac{5}{4}\pi \end{matrix}}$$



$$= \left| x \le 0 \Rightarrow |x| = -x \right| = \iint_D (-x) y^2 \, dx\,dy =$$

$$= \int_\pi^{\frac{5}{4}\pi} \left[\int_1^2 (-r\cos\varphi) \cdot r^2 \sin^2\varphi \cdot r \, dr \right] d\varphi =$$

$$= -\int_\pi^{\frac{5}{4}\pi} \left[\int_1^2 r^4 \sin^2\varphi \cos\varphi \, dr \right] d\varphi = -\int_\pi^{\frac{5}{4}\pi} \left[\frac{r^5}{5} \right]_1^2 \sin^2\varphi \cos\varphi \, d\varphi =$$

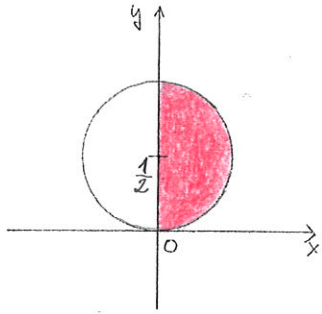
$$= -\int_\pi^{\frac{5}{4}\pi} \left(\frac{32}{5} - \frac{1}{5} \right) \sin^2\varphi \cos\varphi \, d\varphi = -\frac{31}{5} \int_\pi^{\frac{5}{4}\pi} \sin^2\varphi \cos\varphi \, d\varphi =$$

$$= \left| \begin{matrix} t = \sin\varphi \\ dt = \cos\varphi \, d\varphi \end{matrix} \quad \begin{array}{c|c|c} \varphi & \pi & \frac{5}{4}\pi \\ \hline t & 0 & -\frac{\sqrt{2}}{2} \end{array} \right| = \frac{31}{5} \int_{-\frac{\sqrt{2}}{2}}^{0} t^2 \, dt = \frac{31}{5} \left[\frac{t^3}{3} \right]_{-\frac{\sqrt{2}}{2}}^{0} = \frac{31}{\cancel{15}} \cdot \frac{2\sqrt{2}}{\cancel{8}} = \underline{\frac{31\sqrt{2}}{60}}$$

12) $\iint\limits_D \frac{x}{x^2+y^2}\,dx\,dy =$

D: $x^2+y^2 \le y$, $x \ge 0$

$x^2+(y-\frac{1}{2})^2 \le \frac{1}{4}$



$x^2+y^2=y$
$r^2 = r\sin\varphi$
$r = \sin\varphi$

$\boxed{\begin{array}{l} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le \sin\varphi \end{array}}$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^{\sin\varphi} \frac{r\cos\varphi}{r^2}\cdot r\,dr\right]d\varphi =$$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^{\sin\varphi} \cos\varphi\,dr\right]d\varphi = \int_0^{\frac{\pi}{2}} \cos\varphi\cdot[r]_0^{\sin\varphi}\,d\varphi =$$

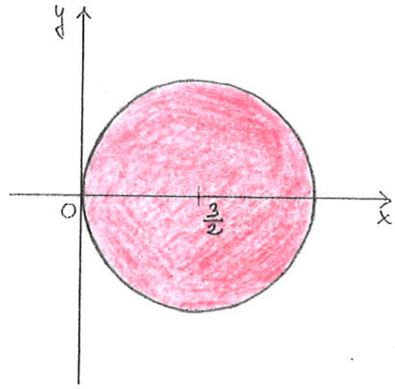
$$= \int_0^{\frac{\pi}{2}} \sin\varphi\cdot\cos\varphi\,d\varphi = \int_0^{\frac{\pi}{2}} \underbrace{\tfrac{1}{2}\sin 2\varphi}_{\frac{1}{2}\cdot 2\sin\varphi\cos\varphi}\,d\varphi =$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2\varphi\right]_0^{\frac{\pi}{2}} = -\frac{1}{4}(\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1}) = -\frac{1}{4}(-1-1) = \underline{\underline{\frac{1}{2}}}$$

13) $\iint\limits_D \sqrt{9-x^2-y^2}\,dx\,dy =$

D: $x^2+y^2 \le 3x$

$(x-\frac{3}{2})^2+y^2 \le \frac{9}{4}$



$x^2+y^2=3x$
$r^2 = 3r\cos\varphi$
$r = 3\cos\varphi$

$\boxed{\begin{array}{l} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 3\cos\varphi \end{array}}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{3\cos\varphi} r\sqrt{9-r^2}\,dr\right]d\varphi =$$

$$= \left|\begin{array}{l} t = 9-r^2 \\ dt = -2r\,dr \end{array} \quad \begin{array}{c|c|l} r & 0 & 3\cos\varphi \\ \hline t & 9 & 9-9\cos^2\varphi = 9\sin^2\varphi \end{array}\right| =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{1}{2}\int_{9\sin^2\varphi}^{9}\sqrt{t}\,dt\right]d\varphi = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{2}{3}t\sqrt{t}\right]_{9\sin^2\varphi}^{9}d\varphi =$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(9\sqrt{9} - 9\sin^2\varphi\sqrt{9\sin^2\varphi}\right)d\varphi =$$

$$= 9\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\varphi\,|\sin\varphi|\,d\varphi\right) = 9\left([\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos^2\varphi)|\sin\varphi|\,d\varphi\right) = \circledast$$

abychom mohli obstranit absolutní hodnotu, musíme rozdělit na dva integrály ① a ②

① $-\frac{\pi}{2} \le \varphi \le 0$: $|\sin\varphi| = -\sin\varphi$

$$-\int_{-\frac{\pi}{2}}^{0}(1-\cos^2\varphi)(-\sin\varphi)\,d\varphi = \left|\begin{array}{l} s = \cos\varphi \\ ds = -\sin\varphi\,d\varphi \end{array} \quad \begin{array}{c|c|c} \varphi & -\frac{\pi}{2} & 0 \\ \hline s & 0 & 1 \end{array}\right| = \int_0^1 (s^2-1)\,ds =$$

$$= \left[\frac{s^3}{3} - s\right]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

② $0 \le \varphi \le \frac{\pi}{2}$: $|\sin\varphi| = \sin\varphi$

$$-\int_0^{\frac{\pi}{2}}(1-\cos^2\varphi)\sin\varphi\,d\varphi = \left|\begin{array}{l} s = \cos\varphi \\ ds = -\sin\varphi\,d\varphi \end{array} \quad \begin{array}{c|c|c} \varphi & 0 & \frac{\pi}{2} \\ \hline s & 1 & 0 \end{array}\right| = \int_0^1 (s^2-1)\,ds =$$

$$= \left[\frac{s^3}{3} - s\right]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$\circledast = 9\left(\frac{\pi}{2} + \frac{\pi}{2} + ① + ②\right) = 9\left(\pi - \frac{4}{3}\right) = \underline{\underline{9\pi - 12}}$$

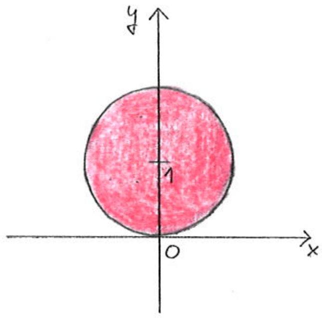
14) $\iint_D |xy|\,dxdy =$ $\qquad D: x^2+y^2-2y \leq 0$

$x^2+(y-1)^2 \leq 1$

$x^2+y^2=2y$

$r^2 = 2r\sin\varphi$

$r = 2\sin\varphi$



$$\boxed{\begin{array}{l} 0 \leq \varphi \leq \pi \\ 0 \leq r \leq 2\sin\varphi \end{array}}$$

$$= \left| y \geq 0 \Rightarrow |y| = y \right| = \iint_D |x| y \,dxdy =$$

$$= \int_0^{\pi} \left[\int_0^{2\sin\varphi} |r\cos\varphi| \cdot r\sin\varphi \cdot r\,dr \right] d\varphi =$$

$$= \int_0^{\pi} \left[\int_0^{2\sin\varphi} r^3 \sin\varphi\, |\cos\varphi|\, dr \right] d\varphi =$$

$$= \int_0^{\pi} \sin\varphi\, |\cos\varphi| \left[\frac{r^4}{4} \right]_0^{2\sin\varphi} d\varphi =$$

$$= 4\int_0^{\pi} \sin^5\varphi\, |\cos\varphi|\, d\varphi = 1 + 2 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

1 $0 \leq \varphi \leq \frac{\pi}{2}$: $|\cos\varphi| = \cos\varphi$

$$4\int_0^{\frac{\pi}{2}} \sin^5\varphi \cos\varphi\, d\varphi = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \end{array} \quad \begin{array}{c|c|c} \varphi & 0 & \frac{\pi}{2} \\ \hline t & 0 & 1 \end{array} \right| = 4\int_0^1 t^5\,dt = 4\left[\frac{t^6}{6} \right]_0^1 = 4 \cdot \frac{1}{6} = \frac{2}{3}$$

2 $\frac{\pi}{2} \leq \varphi \leq \pi$: $|\cos\varphi| = -\cos\varphi$

$$-4\int_{\frac{\pi}{2}}^{\pi} \sin^5\varphi \cos\varphi\, d\varphi = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \end{array} \quad \begin{array}{c|c|c} \varphi & \frac{\pi}{2} & \pi \\ \hline t & 1 & 0 \end{array} \right| = 4\int_0^1 t^5\,dt = 4\left[\frac{t^6}{6} \right]_0^1 = 4 \cdot \frac{1}{6} = \frac{2}{3}$$