

TRANSFORMACE DVOJNÉHO INTEGRÁLU DO POLÁRNÍCH SOUŘADNIC

Obecná transformace:

$$\begin{cases} x = g(u, v) \\ y = h(u, v) \end{cases} \quad \text{transformační rovnice}$$

$$\iint_D f(x, y) dx dy = \iint_{D'} f(g(u, v), h(u, v)) \cdot |J| du dv$$

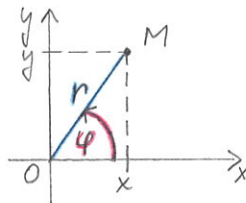
kde J se nazývá jakobián a

$$J = \begin{vmatrix} g'_u & g'_v \\ h'_u & h'_v \end{vmatrix}$$

Účelem transformace je zjednodušit integrační obor nebo integrovanou funkci.

Polární souřadnice

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases}$$



$$M[x, y] \rightarrow M[r, \varphi]$$

$$J = \begin{vmatrix} (r \cos \varphi)'_r & (r \cos \varphi)'_\varphi \\ (r \sin \varphi)'_r & (r \sin \varphi)'_\varphi \end{vmatrix} = \begin{vmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{vmatrix} = r \cos^2 \varphi + r \sin^2 \varphi = r (\underbrace{\cos^2 \varphi + \sin^2 \varphi}_1) = r$$

$$\iint_D f(x, y) dx dy = \iint_{D'} f(r \cos \varphi, r \sin \varphi) \cdot r dr d\varphi$$

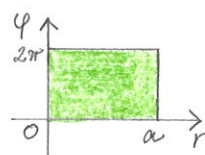
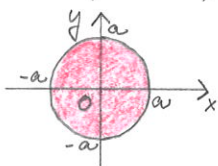
Protože

$$x^2 + y^2 = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi = r^2 (\cos^2 \varphi + \sin^2 \varphi) = r^2$$

budeme výraz $x^2 + y^2$ vždy nahrazovat výrazem r^2 .

Transformaci do polárních souřadnic využijeme, je-li integrační obor ohraničen kružnicí.

$$D: x^2 + y^2 \leq a^2, a > 0 \rightarrow D': \langle 0, a \rangle \times \langle 0, 2\pi \rangle$$



Pomocí transformace do polárních souřadnic jsme kruh převedli na obdélník (dvourozměrný interval).

$$\begin{aligned} -a &\leq x \leq a \\ -\sqrt{a^2 - x^2} &\leq y \leq \sqrt{a^2 - x^2} \end{aligned}$$



$$\begin{aligned} 0 &\leq r \leq a \\ 0 &\leq \varphi \leq 2\pi \end{aligned}$$

Pro správné určení mezí pro proměnné r a φ je třeba množinu D zakreslit.

Vždy musí platit:

1. Rozsah mezí pro proměnnou φ musí být maximálně 2π .
2. Meze pro proměnnou r nesmí být záporné (proměnná r určuje vzdálenost od počátku!).

Pozn.:

- 1, Množina D je ohraničena kružnicí se středem v počátku, tj. kružnicí $x^2 + y^2 = a^2$, $a > 0$, pak jsou meze pro proměnné r i φ perné a nezávisí na pořadí integrace.

- 2, Množina D je ohraničena kružnicí

$$x^2 + y^2 - ax = 0, \quad a > 0 \quad \text{nebo}$$

$$x^2 + y^2 - ay = 0, \quad a > 0$$

$$\rightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

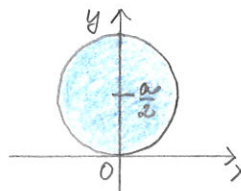
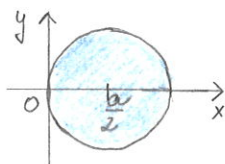
$$\rightarrow x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

$$\rightarrow r^2 - ar \cos \varphi = 0$$

$$\rightarrow r^2 - ar \sin \varphi = 0$$

$$\rightarrow r = a \cos \varphi \quad (*)$$

$$\rightarrow r = a \sin \varphi \quad (*)$$



$$\begin{aligned} -\frac{\pi}{2} &\leq \varphi \leq \frac{\pi}{2} \\ 0 &\leq r \leq a \cos \varphi \end{aligned}$$

$$\begin{aligned} 0 &\leq \varphi \leq \pi \\ 0 &\leq r \leq a \sin \varphi \end{aligned}$$

Perné meze jsou jen pro proměnnou $\varphi \Rightarrow$

\Rightarrow pořadí integrace: 1. podle proměnné r
2. podle proměnné φ

(*) Rovnice dané kružnice v polárních souřadnicích.

Př: Vypočítejte integrály s použitím transformace do polárních souřadnic.

1) $\iint_D (1-2x-3y) dx dy =$

$D: x^2 + y^2 \leq 5$

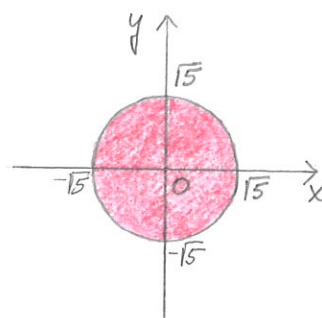
$= \int_0^{\sqrt{5}} \left[\int_0^{2\pi} (1 - \underbrace{2r \cos \varphi}_x - \underbrace{3r \sin \varphi}_y) r d\varphi \right] dr =$

$0 \leq r \leq \sqrt{5}$
 $0 \leq \varphi \leq 2\pi$

$= \int_0^{\sqrt{5}} r [\varphi - 2r \sin \varphi + 3r \cos \varphi]_0^{2\pi} dr =$

$= \int_0^{\sqrt{5}} r (2\pi - \underbrace{2r \sin 2\pi}_0 + \underbrace{3r \cos 2\pi}_1 - 0 + \underbrace{2r \sin 0}_0 - \underbrace{3r \cos 0}_1) dr =$

$= \int_0^{\sqrt{5}} r (2\pi + 3r - 3r) dr = 2\pi \int_0^{\sqrt{5}} r dr = 2\pi \left[\frac{r^2}{2} \right]_0^{\sqrt{5}} = 2\pi \cdot \frac{(\sqrt{5})^2}{2} = \underline{\underline{5\pi}}$



2) $\iint_D y dx dy =$

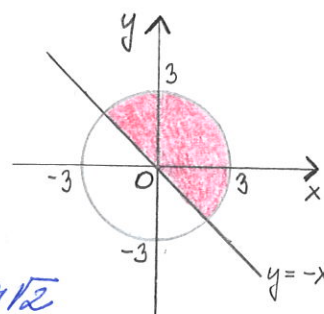
$D: x^2 + y^2 \leq 9, y \geq -x$

$= \int_{-\pi/4}^{3\pi/4} \left[\int_0^3 \underbrace{r^2 \sin \varphi}_y dr \right] d\varphi =$

$0 \leq r \leq 3$
 $-\pi/4 \leq \varphi \leq 3\pi/4$

$= \int_{-\pi/4}^{3\pi/4} \left[\frac{r^3}{3} \right]_0^3 \sin \varphi d\varphi = 9 \int_{-\pi/4}^{3\pi/4} \sin \varphi d\varphi =$

$= 9 [-\cos \varphi]_{-\pi/4}^{3\pi/4} = 9 \left(-\cos \frac{3\pi}{4} + \cos \left(-\frac{\pi}{4} \right) \right) = 9 \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \underline{\underline{9\sqrt{2}}}$



3) $\iint_D \sqrt{1-x^2-y^2} dx dy =$

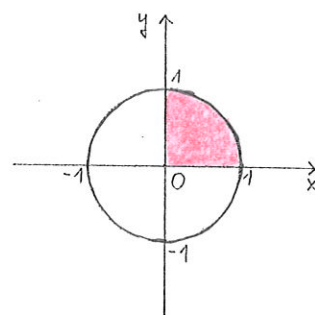
$D: x^2 + y^2 \leq 1, x \geq 0, y \geq 0$

$= \int_0^1 \left[\int_0^{\pi/2} r \sqrt{1-r^2} d\varphi \right] dr = \int_0^1 r \sqrt{1-r^2} [\varphi]_0^{\pi/2} dr =$

$0 \leq r \leq 1$
 $0 \leq \varphi \leq \frac{\pi}{2}$

$= \frac{\pi}{2} \int_0^1 r \sqrt{1-r^2} dr = \left| \begin{array}{l} t = 1-r^2 \\ dt = -2r dr \end{array} \right| \frac{r}{t} \left| \begin{array}{l} 0 \\ 1 \end{array} \right| \frac{1}{0} =$

$= \frac{\pi}{4} \int_0^1 \sqrt{t} dt = \frac{\pi}{4} \left[\frac{2}{3} t^{3/2} \right]_0^1 = \underline{\underline{\frac{\pi}{6}}}$



$$4) \iint_D \frac{\ln(x^2+y^2)}{x^2+y^2} dx dy =$$

$$D: 1 \leq x^2 + y^2 \leq e^2$$

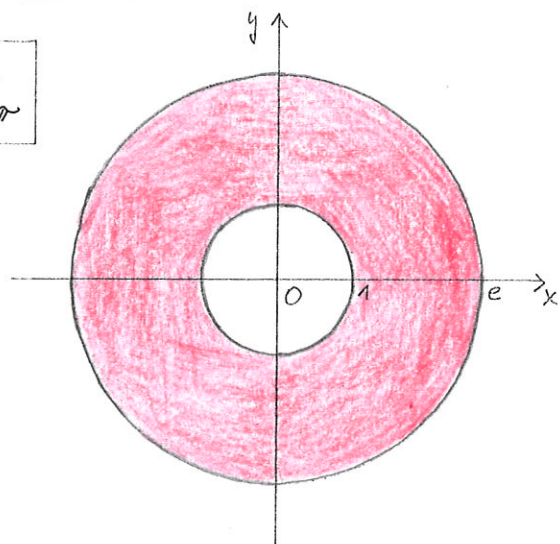
$$\begin{cases} 1 \leq r \leq e \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$= \int_1^e \left[\int_0^{2\pi} \frac{\ln r^2}{r^2} \cdot r d\varphi \right] dr = \int_1^e \left[\int_0^{2\pi} \frac{2 \ln r}{r} d\varphi \right] dr =$$

$$= \int_1^e \frac{2 \ln r}{r} [\varphi]_0^{2\pi} dr = 4\pi \int_1^e \frac{\ln r}{r} dr =$$

$$= \left| \begin{matrix} t = \ln r \\ dt = \frac{1}{r} dr \end{matrix} \right| \frac{r}{t} \bigg|_0^1 = 4\pi \int_0^1 t dt =$$

$$= 2\pi [t^2]_0^1 = \underline{\underline{2\pi}}$$



$$5) \iint_D (x^2 + y^2) dx dy =$$

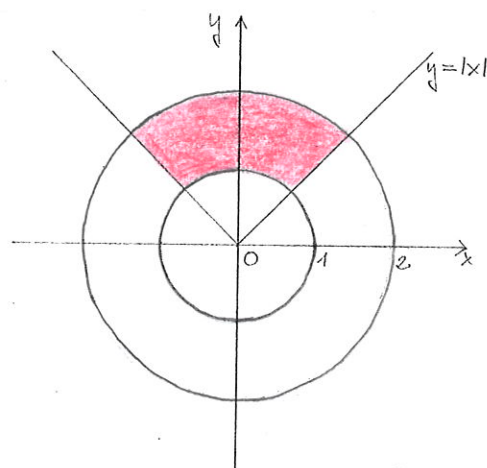
$$D: 1 \leq x^2 + y^2 \leq 4, y \geq |x|$$

$$\begin{cases} 1 \leq r \leq 2 \\ \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4} \end{cases}$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\int_1^2 r^3 dr \right] d\varphi = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\frac{1}{4} r^4 \right]_1^2 d\varphi =$$

$$= \left(4 - \frac{1}{4} \right) \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi = \frac{15}{4} [\varphi]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} =$$

$$= \frac{15}{4} \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) = \frac{15}{4} \cdot \frac{\pi}{2} = \underline{\underline{\frac{15\pi}{8}}}$$



$$6) \iint_D \arctan \frac{x}{y} dx dy =$$

$$D: x^2 + y^2 \geq 1, x^2 + y^2 \leq 9, y \geq \frac{x}{\sqrt{3}}, y \leq x\sqrt{3}$$

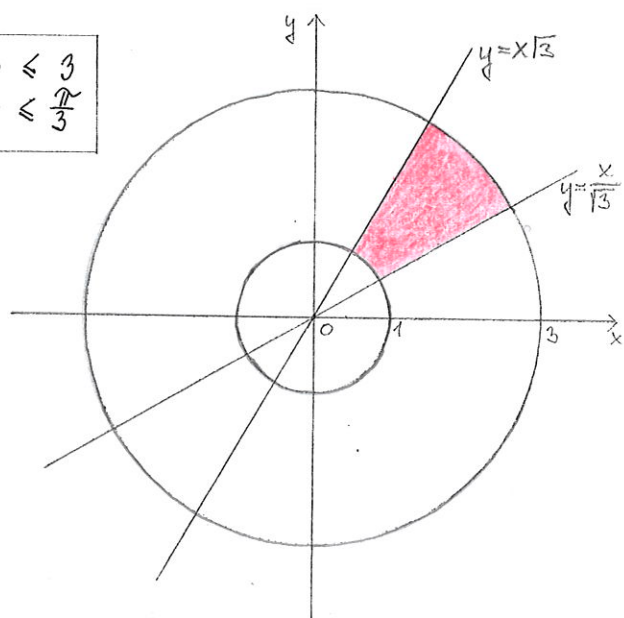
$$\begin{cases} 1 \leq r \leq 3 \\ \frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3} \end{cases}$$

$$= \int_1^3 \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \cdot \arctan \frac{r \sin \varphi}{r \cos \varphi} d\varphi \right] dr =$$

$$= \int_1^3 \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \cdot \arctan(\tan \varphi) d\varphi \right] dr = \int_1^3 \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \varphi d\varphi \right] dr =$$

$$= \int_1^3 r \left[\frac{1}{2} \varphi^2 \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} dr = \frac{1}{2} \left(\frac{\pi^2}{9} - \frac{\pi^2}{36} \right) \int_1^3 r dr = \frac{1}{2} \cdot \frac{3\pi^2}{36} \int_1^3 r dr =$$

$$= \frac{\pi^2}{24} \left[\frac{1}{2} r^2 \right]_1^3 = \frac{\pi^2}{48} \cdot 8 = \underline{\underline{\frac{\pi^2}{6}}}$$

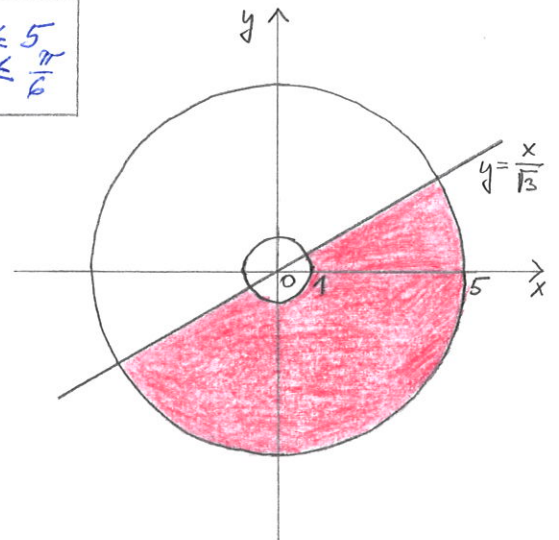


$$7) \iint_D \frac{x}{\sqrt{x^2+y^2}} dx dy =$$

$$D: 1 \leq x^2 + y^2 \leq 25, \quad y \leq \frac{x}{\sqrt{3}}$$

$$\begin{aligned} &= \int_1^5 \left[\int_{-\frac{5\pi}{6}}^{\frac{\pi}{6}} \frac{r \cos \varphi}{\sqrt{r^2}} \cdot r d\varphi \right] dr = \\ &= \int_1^5 \left[\int_{-\frac{5\pi}{6}}^{\frac{\pi}{6}} r \cos \varphi d\varphi \right] dr = \int_1^5 r \left[\sin \varphi \right]_{-\frac{5\pi}{6}}^{\frac{\pi}{6}} dr = \\ &= \int_1^5 r \left(\underbrace{\sin \frac{\pi}{6}}_{\frac{1}{2}} - \underbrace{\sin(-\frac{5\pi}{6})}_{-\frac{1}{2}} \right) dr = \\ &= \int_1^5 r \left(\frac{1}{2} + \frac{1}{2} \right) dr = \int_1^5 r dr = \left[\frac{r^2}{2} \right]_1^5 = \\ &= \frac{25}{2} - \frac{1}{2} = \frac{24}{2} = \underline{\underline{12}} \end{aligned}$$

$$\begin{aligned} 1 \leq r \leq 5 \\ -\frac{5\pi}{6} \leq \varphi \leq \frac{\pi}{6} \end{aligned}$$

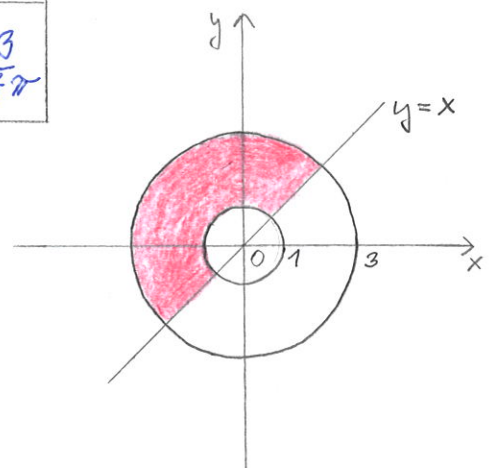


$$8) \iint_D e^{-x^2-y^2} dx dy =$$

$$D: 1 \leq x^2 + y^2 \leq 9, \quad y \geq x$$

$$\begin{aligned} &= \int_1^3 \left[\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} r \cdot e^{-r^2} d\varphi \right] dr = \int_1^3 r e^{-r^2} \left[\varphi \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} dr = \\ &= \int_1^3 r e^{-r^2} \left(\frac{5\pi}{4} - \frac{\pi}{4} \right) dr = \pi \int_1^3 r e^{-r^2} dr = \\ &= \left| \begin{array}{l} t = -r^2 \\ dt = -2r dr \\ -\frac{1}{2} dt = r dr \end{array} \right| \quad \left| \begin{array}{c|c|c} r & 1 & 3 \\ t & -1 & -9 \end{array} \right| = \\ &= -\frac{1}{2} \pi \int_{-1}^{-9} e^t dt = \frac{\pi}{2} \int_{-9}^{-1} e^t dt = \\ &= \frac{\pi}{2} \left[e^t \right]_{-9}^{-1} = \underline{\underline{\frac{\pi}{2} (e^{-1} - e^{-9})}} \end{aligned}$$

$$\begin{aligned} 1 \leq r \leq 3 \\ \frac{\pi}{4} \leq \varphi \leq \frac{5\pi}{4} \end{aligned}$$

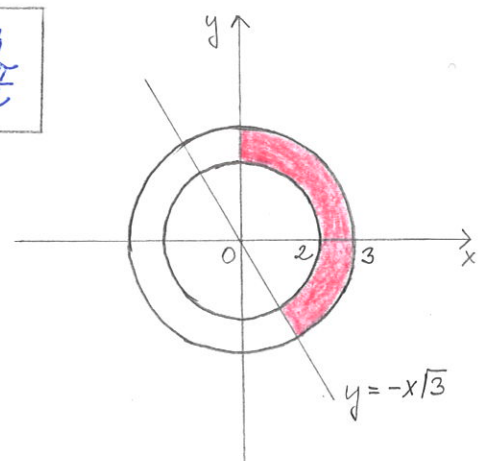


$$9) \iint_D \frac{y}{x^2+y^2} dx dy =$$

$$D: 4 \leq x^2 + y^2 \leq 9, \quad y \geq -x\sqrt{3}, \quad x \geq 0$$

$$\begin{aligned} &= \int_2^3 \left[\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{r \sin \varphi}{r^2} \cdot r d\varphi \right] dr = \\ &= \int_2^3 \left[\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \varphi d\varphi \right] dr = \int_2^3 \left[-\cos \varphi \right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}} dr = \\ &= \int_2^3 \left(\underbrace{-\cos \frac{\pi}{2}}_0 + \underbrace{\cos(-\frac{\pi}{3})}_{\frac{1}{2}} \right) dr = \frac{1}{2} \int_2^3 dr = \\ &= \frac{1}{2} \left[r \right]_2^3 = \frac{1}{2} (3 - 2) = \underline{\underline{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned} 2 \leq r \leq 3 \\ -\frac{\pi}{3} \leq \varphi \leq \frac{\pi}{2} \end{aligned}$$



$$10) \iint_D \sin \sqrt{x^2+y^2} dx dy$$

$$= \int_{\pi}^{2\pi} \left[\int_0^{\pi} r \sin r dr \right] dr =$$

$$= \int_{\pi}^{2\pi} r \sin r [\varphi]_0^{\pi} dr = \pi \int_{\pi}^{2\pi} r \sin r dr =$$

$$= \left| \begin{array}{l} u = r \quad v' = \sin r \\ u' = 1 \quad v = -\cos r \end{array} \right| =$$

$$= \pi ([-r \cos r]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos r dr) =$$

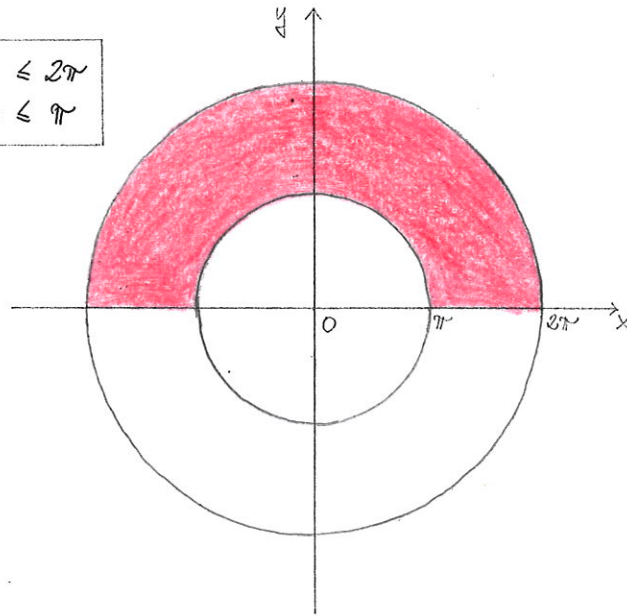
$$= \pi ([-r \cos r]_{\pi}^{2\pi} + [\sin r]_{\pi}^{2\pi}) =$$

$$= \pi (-2\pi \underbrace{\cos 2\pi}_1 + \pi \underbrace{\cos \pi}_{-1} + \underbrace{\sin 2\pi}_0 - \underbrace{\sin \pi}_0) =$$

$$= \pi (-2\pi - \pi) = \underline{\underline{-3\pi^2}}$$

$$D: \pi^2 \leq x^2+y^2 \leq 4\pi^2, y \geq 0$$

$$\begin{array}{l} \pi \leq r \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{array}$$



$$11) \iint_D |x| y^2 dx dy$$

$$= |x \leq 0 \Rightarrow |x| = -x| = \iint_D (-x) y^2 dx dy =$$

$$= \int_{\pi}^{\frac{5}{4}\pi} \left[\int_1^2 (-r \cos \varphi) \cdot r^2 \sin^2 \varphi \cdot r dr \right] d\varphi =$$

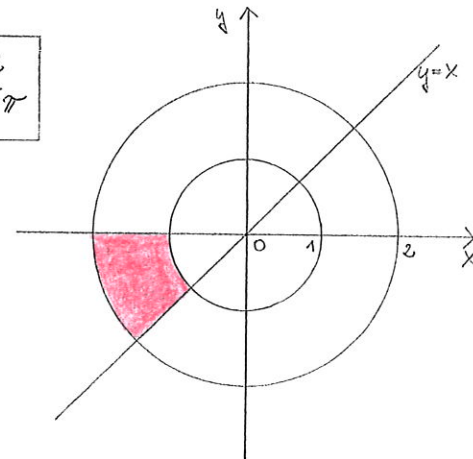
$$= - \int_{\pi}^{\frac{5}{4}\pi} \left[\int_1^2 r^4 \sin^2 \varphi \cos \varphi dr \right] d\varphi = - \int_{\pi}^{\frac{5}{4}\pi} \left[\frac{r^5}{5} \right]_1^2 \sin^2 \varphi \cos \varphi d\varphi =$$

$$= - \int_{\pi}^{\frac{5}{4}\pi} \left(\frac{32}{5} - \frac{1}{5} \right) \sin^2 \varphi \cos \varphi d\varphi = - \frac{31}{5} \int_{\pi}^{\frac{5}{4}\pi} \sin^2 \varphi \cos \varphi d\varphi =$$

$$= \left| \begin{array}{l} t = \sin \varphi \\ dt = \cos \varphi d\varphi \end{array} \right| \begin{array}{l} \varphi | \pi | \frac{5}{4}\pi \\ t | 0 | -\frac{\sqrt{2}}{2} \end{array} = \frac{31}{5} \int_{-\frac{\sqrt{2}}{2}}^0 t^2 dt = \frac{31}{5} \left[\frac{t^3}{3} \right]_{-\frac{\sqrt{2}}{2}}^0 = \frac{31}{5} \cdot \frac{2\sqrt{2}}{8} = \underline{\underline{\frac{31\sqrt{2}}{20}}}$$

$$D: 1 \leq x^2+y^2 \leq 4, x \leq y \leq 0$$

$$\begin{array}{l} 1 \leq r \leq 2 \\ \pi \leq \varphi \leq \frac{5}{4}\pi \end{array}$$



$$12) \iint_D \frac{x}{x^2+y^2} dx dy =$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin \varphi} \frac{r \cos \varphi}{r^2} \cdot r dr d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^{\sin \varphi} \cos \varphi dr \right] d\varphi = \int_0^{\frac{\pi}{2}} \cos \varphi \cdot [r]_0^{\sin \varphi} d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \sin \varphi \cdot \cos \varphi d\varphi = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2\varphi d\varphi =$$

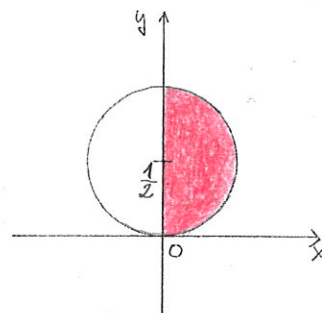
$$= \frac{1}{2} \left[-\frac{1}{2} \cos 2\varphi \right]_0^{\frac{\pi}{2}} = -\frac{1}{4} (\underbrace{\cos \pi}_{-1} - \underbrace{\cos 0}_1) = -\frac{1}{4} (-1-1) = \underline{\underline{\frac{1}{2}}}$$

$$D: x^2+y^2 \leq y, \quad x \geq 0$$

$$x^2 + (y-\frac{1}{2})^2 \leq \frac{1}{4}$$

$$\begin{aligned} x^2+y^2 &= y \\ r^2 &= r \sin \varphi \\ r &= \sin \varphi \end{aligned}$$

$$\begin{aligned} 0 &\leq \varphi \leq \frac{\pi}{2} \\ 0 &\leq r \leq \sin \varphi \end{aligned}$$



$$13) \iint_D \sqrt{9-x^2-y^2} dx dy =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{3 \cos \varphi} r \sqrt{9-r^2} dr \right] d\varphi =$$

$$= \left| \begin{array}{c|c|c} t=9-r^2 & r|0 & 3 \cos \varphi \\ dt=-2r dr & t|9 & 9-9 \cos^2 \varphi = 9 \sin^2 \varphi \end{array} \right| =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{2} \int_0^9 \sqrt{t} dt \right]_{9 \sin^2 \varphi}^9 d\varphi = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3} t \sqrt{t} \right]_{9 \sin^2 \varphi}^9 d\varphi =$$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (9\sqrt{9} - 9 \sin^2 \varphi \sqrt{9 \sin^2 \varphi}) d\varphi =$$

$$= 9 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi |\sin \varphi| d\varphi \right) = 9 \left(\left[\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 \varphi) |\sin \varphi| d\varphi \right) = (*)$$

$$① -\frac{\pi}{2} \leq \varphi \leq 0 : |\sin \varphi| = -\sin \varphi$$

$$\begin{aligned} -\int_{-\frac{\pi}{2}}^0 (1-\cos^2 \varphi) (-\sin \varphi) d\varphi &= \left| \begin{array}{c|c|c} s = \cos \varphi & s|0 & 1 \\ ds = -\sin \varphi d\varphi & s|1 & 0 \end{array} \right| = \int_0^1 (s^2-1) ds = \\ &= \left[\frac{s^3}{3} - s \right]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3} \end{aligned}$$

$$② 0 \leq \varphi \leq \frac{\pi}{2} : |\sin \varphi| = \sin \varphi$$

$$\begin{aligned} -\int_0^{\frac{\pi}{2}} (1-\cos^2 \varphi) \sin \varphi d\varphi &= \left| \begin{array}{c|c|c} s = \cos \varphi & s|0 & \frac{\pi}{2} \\ ds = -\sin \varphi d\varphi & s|1 & 0 \end{array} \right| = \int_0^1 (s^2-1) ds = \\ &= \left[\frac{s^3}{3} - s \right]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3} \end{aligned}$$

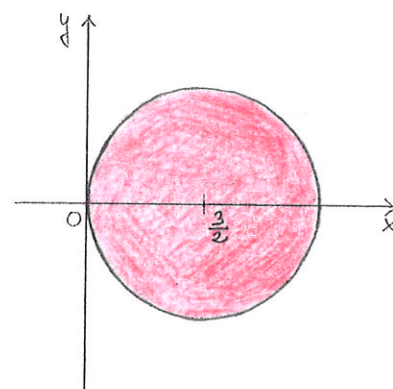
$$(*) = 9 \left(\frac{\pi}{2} + \frac{\pi}{2} + ① + ② \right) = 9 \left(\pi - \frac{4}{3} \right) = \underline{\underline{9\pi - 12}}$$

$$D: x^2+y^2 \leq 3x$$

$$(x-\frac{3}{2})^2 + y^2 \leq \frac{9}{4}$$

$$\begin{aligned} x^2+y^2 &= 3x \\ r^2 &= 3r \cos \varphi \\ r &= 3 \cos \varphi \end{aligned}$$

$$\begin{aligned} -\frac{\pi}{2} &\leq \varphi \leq \frac{\pi}{2} \\ 0 &\leq r \leq 3 \cos \varphi \end{aligned}$$



abychom mohli odstranit absolutnu hodnotu, musime rozdelit na dva integraly ① a ②

$$14) \iint_D |xy| dx dy =$$

$$= |y \geq 0 \Rightarrow |y| = y| = \iint_D |x| y dx dy =$$

$$= \int_0^{\pi} \left[\int_0^{2\sin\varphi} |r\cos\varphi| \cdot r\sin\varphi \cdot r dr \right] d\varphi =$$

$$= \int_0^{\pi} \left[\int_0^{2\sin\varphi} r^3 \sin\varphi |\cos\varphi| dr \right] d\varphi =$$

$$= \int_0^{\pi} \sin\varphi |\cos\varphi| \left[\frac{r^4}{4} \right]_0^{2\sin\varphi} d\varphi =$$

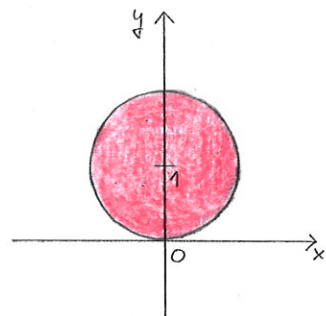
$$= 4 \int_0^{\pi} \sin^5\varphi |\cos\varphi| d\varphi = \textcircled{1} + \textcircled{2} = \frac{2}{3} + \frac{2}{3} = \underline{\underline{\frac{4}{3}}}$$

$$D: x^2 + y^2 - 2y \leq 0$$

$$x^2 + (y-1)^2 \leq 1$$

$$\begin{aligned} x^2 + y^2 &= 2y \\ r^2 &= 2r\sin\varphi \\ r &= 2\sin\varphi \end{aligned}$$

$$\begin{aligned} 0 &\leq \varphi \leq \pi \\ 0 &\leq r \leq 2\sin\varphi \end{aligned}$$



$$\textcircled{1} 0 \leq \varphi \leq \frac{\pi}{2}: |\cos\varphi| = \cos\varphi$$

$$4 \int_0^{\frac{\pi}{2}} \sin^5\varphi \cos\varphi d\varphi = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi d\varphi \end{array} \right.$$

$$\frac{\varphi}{t} \left| \begin{array}{l} 0 \\ 0 \end{array} \right| \frac{\frac{\pi}{2}}{1} = 4 \int_0^1 t^5 dt = 4 \left[\frac{t^6}{6} \right]_0^1 = 4 \cdot \frac{1}{6} = \frac{2}{3}$$

$$\textcircled{2} \frac{\pi}{2} \leq \varphi \leq \pi: |\cos\varphi| = -\cos\varphi$$

$$-4 \int_{\frac{\pi}{2}}^{\pi} \sin^5\varphi \cos\varphi d\varphi = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi d\varphi \end{array} \right.$$

$$\frac{\varphi}{t} \left| \begin{array}{l} \frac{\pi}{2} \\ 1 \end{array} \right| \frac{\pi}{0} = 4 \int_1^0 t^5 dt = 4 \left[\frac{t^6}{6} \right]_1^0 = 4 \cdot \frac{1}{6} = \frac{2}{3}$$