Hypergroupoids on Special Partially Ordered Sets

Josef Zapletal

European Polytechnical Institute, LLC.
686 04 Kunovice, Czech Republic,
e-mail: zapletal@vos.cz

Abstract. In recitals of this paper there we will be repeated the definitions and main results mentioned 1n [15] without proofs. The definition of the multioperation on partially ordered carrier set in this papers is idempotent, commutative but not associative operation. In the opening part of this article there we repeat fundamental definitions and some theorems without proofs. In the next part of this conference paper we introduce the conceptions of distinguishing, weakly distinguishing subsets and the concept of β hypergroupoid. Finally some properties of this notions on multigroupoids with a concrete special ordering of carrier sets are studied..

Key words. Hypersemigroup, binary hyperoperation, hypergroupoid, distinguishing subset, weakly distinguishing subset,hypergroupoid with the property β. Covering and maximal β covering of hypergroupoid.

1 Introduction

1.1 Definition *A hypergroupoid* (or *a multigroupoid*) is a pair $(M, *)$ where M is a nonempty set and $* : M \times M \rightarrow \mathcal{P}^*(M)$ is a binary hyperoperation called also a multioperation. ($\mathcal{P}^*(M)$ is the system of all nonempty subsets of M).
A semihypergroup is an associative hypergrupoid, i.e. hypergrupoid satisfying the equality $(a * b) * c = a * (b * c)$ for every triad a, b, $c \in M$.

1.2 Introduction We denote by $\mathcal{M}$ a partially ordered set M with the ordering $\leq$ and with the greatest element I which will be inscribed in the next part of this article with $\mathcal{M} = (M, \leq, I)$

1.3 Definition Let $x_i \in M \mid i \in J$ where J is an index set. By the length r of a chain we understand the circumstances that the chain consists from $r+1$ elements of the set M and is of the form

$$x_0 \prec x_1 \prec x_2 \prec \ \ldots \prec x_r \qquad [x_0, x_r]$$

(where the notation $x_i \prec x_{i+1}$ means that the element x_i is covered by the element x_{i+1} it is $x_i < x_{i+1}$ and does not exist $x \in M$ for which $x_i < x < x_{i+1}$. We define the length of the partially ordered set ($\mathcal{M} = (M, \leq, I)$ as

$$max\{r_j \mid \text{where } r_j, j \in J \text{ are lengths of chains in } M\}.$$

We suppose the partially ordered sets of finite length.

1.4 Definition We define for arbitrary $x, y \in M$ on $\mathcal{M} = (M, \leq, I)$ the binary hyperoperation $\circ$ as follows:

$$x \circ y = \{ \min (\mathcal{X} \cap \mathcal{Y})\}.$$

Where $\mathcal{X} = \{m_j \mid m_j \in M, x \leq m_j\}$ for all j from index set J and similarly the set $\mathcal{Y} = \{m_k \mid m_k \in M, y \leq m_k\}$ for all k from index set K. We inscribe then the set $\mathcal{M}$ with such defined binary operation with $\mathcal{M} = (M \leq, \circ, I)$.

1.5 Lemma The hyperoperation of multiplication $\circ$ on $\mathcal{M} = (M \leq, \circ, I)$ is idempotent.

1.6 Lemma The binary hyperoperation $\circ$ on $\mathcal{M} = (M \leq, \circ, I)$ is commutative.

1.7 Theorem $\mathcal{M} = (M, \leq, \circ, I)$ is commutative hypergroupoid.

1.8 Theorem Let us suppose that the operation $\circ$ is single-valued for all the elements of M. Then $\circ$ is a semilattice operation of supremum and $\mathcal{M} = (M, \leq, \circ, I)$ is an upper semilattice.

Proof. The first affirmation follows from the definition of the multioperation. $\circ$ for $x \circ y = \{ \min (\mathcal{X} \cap \mathcal{Y})\}$. where $\mathcal{X}, \mathcal{Y}$ are dual ideals of the elements x, y. (See 1.4). The operation $\circ$ satisfies idempotency and commutativity (1.6, 1.7). From the unicity of the operation there follows the associativity likewise. Hence $\circ = \vee$ where $\vee$ is a semilattice operation.

1.9 Remark The ordering of the carrier set M characterizes many properties of the hypergroupoid $\mathcal{M} = (M, \leq, \circ, I)$.

2. Some constructions on hypergroupoids with special kinds of carrier sets .

2.1 Definition Let $M, \leq, I$ be a finite partly ordered set with the the ordering $\leq$ and the greatest element I. Let a, b be a pair of elements of M such that $a \leq b$ Then we define as the interval bounded by the elements a and b and denote by $[a, b]$ the set of all the elements x of M for which $a \leq x \leq b$ holds.

2.2 Definition Let $M, \leq, \circ I$ be a finite partly ordered multigroupoid satisfying the Jordan-Dedekind chain condition where the relation of partly ordering is defined as follows:

$$x \leq y \text{ for all } x \text{ and } y \text{ for which } d(y) = d(x) - 1 \text{ and } x \parallel y \text{ if } d(x) = d(y).$$

Hence x is in the relation $\leq$ with all its descendants. (See Fig.1)

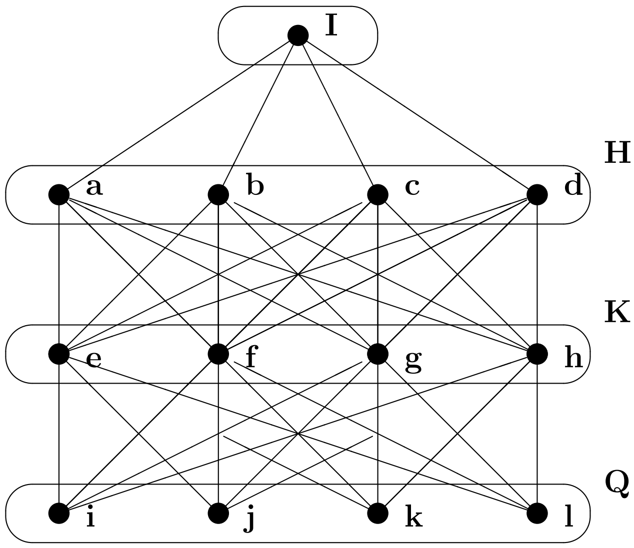

Figure 1

2.3 Definition Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid $L \subset M, u \in M$. We say that the elements $x, y \in M, x \neq y$ are distinguished by u with respect to L if $u \circ x \subset L$ and $u \circ y \not\subset L$ or $u \circ x \not\subset L$ and $u \circ y \subset L$.
We say that L distinguishes $\mathcal{M}(f)$ if for each $x, y \in M, x \neq y$, there is $u \in M$ such that x, y are distinguished by u with respect to L.

2.4 Example Let the partially ordering of the set $M = \{I, a, b, c, d, e, f, g, h, i, j, k, l, m, \}$. Which is given by the graph on the Figure 1. and let the operation $x \circ y$ where x, y are arbitrary elements in M is described by knots lying over the double x, y which are connected by edges with them. In the next we will study its the subsets I, H, K, Q according two relation of distinguishing.

2.5 Affirmation Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypersemigroupoid given in 2.4 and $L = I \cup K \subset M$. Then L distinguishes $\mathcal{M}(M, \leq, \circ, I)$.
Proof. Let x, y are arbitrary different elements in M.

a) Let $p(x) = p(y)$. Then it is sufficient to put $u = x$ and after that $p(u \circ x) = p(x)$ and $p(u \circ y) = p(y) - 1$. This implies that $u \circ x \subset L$ and $u \circ y \not\subset L$ or conversely.

b) Let $p(x) \neq p(y)$. Without generality loss we can suppose $p(x) < p(y)$. Then if the

element $x \in L$ and the element $y \notin L$ it is sufficient to put $u = x$ and $u \circ x = x \in L$ and $u \circ y = y \notin L$. Let us suppose that both elements lie in L. The situation that non of the elements x, y does not lie in L would be analogous. In this case it is sufficient to choose arbitrary element z for which $p(z) = p(y)$ and than $z \circ x = z \in L$ and $p(z \circ y) = p(z) - 1$ and $z \circ y \notin L$.
We chose for arbitrary double x, y an element u which distinguishes this double with respect to $L = I \cup K$.

2.6 Definition Let $M, \leq, I$ be a finite partly ordered set with the greatest element I. We denote the set of its dual atoms by D. Further we denote by $D(x) \subseteq D$ the subset containing all elements $d_j \in D, j \in J$ where J is the index set, for which $x \leq d_j$.

2.7 Definition Let $M, \leq, I$ be a finite partly ordered set with the greatest element I satisfying Jordan-Dedekind Chain condition. We denote by the depth $d(a)$ of the element $a \in M$ the length of the interval $[a, I]$ (especially $d(I) = 0$). We say that the element y covers the element x and we write $x \prec$ iff $x \leq y$ and $d(y) = d(x) - 1$.

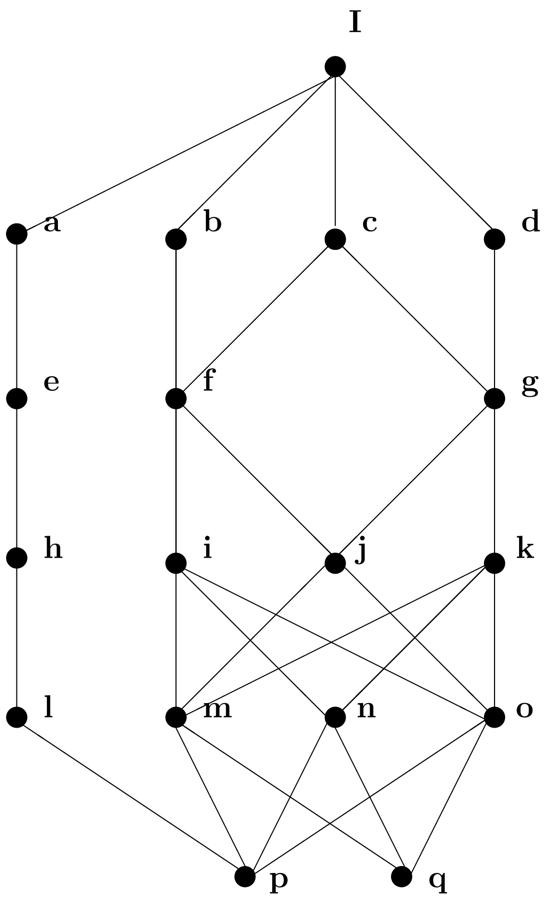

Figure 2

2.8 Definition Let $M, \leq, I$ be a finite partly ordered set with the greatest element I. We say that M satisfies the condition β if for arbitrary two elements $x, y \in M$ the following relation $\{D(x) \cap D(y) - \{I\}\} \neq \emptyset$ holds. We call then $\mathcal{M} = (M, \leq, \circ, I)$ the β hypergroupoid.

2.9 Lemma Let $\mathcal{M} = (M, \leq, \circ, I)$ be a β hypergroupoid. Then the one-point subset $\{I\}$ and the set D distinguish M.

Proof. Let x, y are arbitrary elements in M. From the property β then follows the existence of at least one dual atom $d \in D$ for which either $x < d$ and $y \parallel d$ or $x \parallel d$ and $y < d$. then either $d \circ x = d \neq I$ and $d \circ y = I$ or $d \circ x = I$ and $d \circ y = d \neq I$. Hence both subsets $\{I\}$, D distinguish M.

2.10 Construction Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid. We will construct a maximal β covering $\boldsymbol{\mathcal{CM}}$ of this hypergroupoid by subhypergroupoids which have the property β. We denote them by $\boldsymbol{\mathcal{C}}M_j$

Construction. We begin from the maximal element. The first subhypergroupoid will contain all the dual-atoms at first and thereafter all the elements which are forgoers at least of two dual-atoms. The elements of such constructed subhypergroupoid which is dual upper ideal can be distinguished with respect to the maximal element I. After this we continue such that we add to the carrier of this first subhypergroupoid all predecessors with the property β. The set M is finite. Then there are two possibilities.

1) Al the elements of the carrier set M are earmarked (chosen). In this case the covering $\boldsymbol{\mathcal{CM}}$ has exactly one suphypergroupoid and it is the given hypergroupoid $\mathcal{M} = (M, \leq, \circ, I)$ itself.

2) There are no further elements for which the further extension of the subhypergroupoid will fulfil the property beta. We symbolize chosen subhypergroupoid by $\boldsymbol{\mathcal{C}}M_1$. We choose arbitrary maximal element sub elements of $\boldsymbol{\mathcal{C}}M_1$ and we denote it by I_2. We assign to this element I_2 all its predecessors as dual atoms of new partially ordered set of the second subhypergroupoid. It may be that are no predecessors of I_2. Then the set carrier of $\boldsymbol{\mathcal{C}}M_2$ is only one-point set. If such predecessors exist, we create the second maximal β subhypergroupoid as well as in the case of $\boldsymbol{\mathcal{C}}M_1$. The next problem can tur up at the construction of the third and further subhypergroupoids. By selection of the predecessors for the third and further maximal elements as dual atoms we suppose only those which were not exhausted to some previous maximal element. After a fine number of steps the construction of the maximal β covering $\boldsymbol{\mathcal{CM}}$ is finished. The construction is not single-valued.This can be seen in the next example.

2.11 Example Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid which operation is given at the figure two. We construct a covering of this hypergroupoid $\boldsymbol{\mathcal{CM}}$ by subhypergroupoids which we denote by $\boldsymbol{\mathcal{C}}M_i$

Construction. We begin from the maximal element. The first subhypergroupoid contains all the dual-atoms at first and thereafter all the elements which are forgoers at least of two dual-atoms. After this we continue such that we add to the carrier of this first subhypergroupoid which have over them with respect to the ordering $\leq$ at least two elements. With respect to finality we finish either at that moment when no element of the given hypergroupoid satisfies this condition or all the element are choice into the subhypergroupoid. In our case the first subhypergroupid $\mathcal{C}M_1$ is created by the set $\{I, a, b, c, d, f, g, j\}$. The next subhypergroupoids are constructed such that their maximal elements are the irreducible forgoers of the absorbed elements of $\mathcal{C}M_1$. We take the forgoer of the element a it is the element e and its dual atom with respect to the ordering $\leq$ is the element h. The element e has no other dual atoms and hence no forgoers which would be forgoers of other dual atoms. Such the second subhypergroupoid $\mathcal{C}M_2$ creates the set $\{e, h\}$. The further irreducible elements are l, i, k. The sentence of the choice features forms of the particular subhypergroupoids. We have three possibilities of choice, we do all of them.

Let e be the maximal element of the third subhypergroupid. This one has only one dual atom and it is the element p. Such $\mathcal{C}M_3$ contains only two elements l, p.

The following maximal element can be i. Its dual atoms are the elements m, n, o and their only non-exhausted sequential element is q and hence the carrier set of the fourth subhypergroupoid is $\{i, m, n, o, q\}$

Finally the last element k forms also the last subhypergroupoid $\mathcal{C}M_5$ has only one point set $\{k\}$.

The other ordering of the further irreducible elements can be i, e, k. When we create the superhypergroupids we obtain as the set carrier of $\mathcal{C}M_3$ the set $\{i, m, no, p, q\}$ and $\mathcal{C}M_4$ and $\mathcal{C}M_5$ are one point superhypergroupids, in the concrete they are created by one element sets $\{e\}$ and $\{k\}$.

2.12 Definition Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid,$L \subset M$ we say that the subset L weakly distinguishes the set M when for every two different elements $x, y \in M$ there exists an element $u \in M$ for which either $u \circ x \cap L \neq \emptyset$ and $u \circ y \not\subseteq L$ or $u \circ x \not\subseteq L$ and $u \circ x \cap L \neq \emptyset$.

2.13 Affirmation Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid which operation is given by the ordering of carrier set at the figure two. Let $\mathcal{CM}$ be a disjunctive covering of the set M. We construct the set L such that we choose from every $\mathcal{C}M_j \in \mathcal{CM}$ its greatest element. Then the subset L weakly distinguishes the given hypergroupoid $\mathcal{M}$.

Proof. The construction of the covering is not single-valued. In the next we suppose these $\mathcal{C}M_i$:
1) $\mathcal{C}M_1 = \{I, a, b, c, d, f, g, j\}$,
2) $\mathcal{C}M_2 = \{e, h\}$,
3) $\mathcal{C}M_3 = \{i, m, n, o, pq\}$,
4) $\mathcal{C}M_4 = \{k\}$,
5) $\mathcal{C}M_4 = \{l\}$.

Then the weakly distinguishing set for this covering is $L = \{I, e, i, k, l\}$.

We prove the affirmation such that we find and print into the table for every two elements $x, y \in M$ one element $u \in M$ for which either $((u \circ x) \cap L \neq \emptyset) \wedge (u \circ y \notin L)$ or $(u \circ x \notin L) \wedge ((u \circ y) \cap L \neq \emptyset)$. For u printed as bold face it concludes only to weakly-distinguishing by L.

	I	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
I		a	b	c	d	a	f	g	a	b	b	g	h	b	b	b	b	b
a			a	a	a	e	f	g	e	b	b	a	l	b	b	b	b	b
b				b	b	b	c	c	a	m	c	c	a	c	c	c	c	c
c					f	a	b	d	a	b	b	d	a	b	b	b	b	b
d						a	b	c	a	b	b	b	a	a	b	b	b	b
e							a	b	l	b	b	d	h	b	b	b	b	b
f								b	a	m	d	b	a	d	d	d	d	d
g									c	m	b	m	a	a	a	a	a	a
h										a	a	a	p	a	a	a	b	b
i											d	b	a	p	p	p	a	m
j												m	a	p	p	n	a	p
k													a	b	b	q	o	a
l														a	a	a	b	a
m															**n**	**m**	a	**n**
n																**n**	a	**c**
o																	a	**m**
p																		a
q																		

Table 1

2.14 Theorem Let $\mathcal{M} = (M, \leq, \circ, I)$ be a hypergroupoid, $\boldsymbol{\mathcal{CM}}$ arbitrary covering of $\mathcal{M}$ constructed by the construction 2.10. We denote the maximal element of $\boldsymbol{\mathcal{C}}M_j \mid j \in J$ where J is the index set as Ij. Let $L = \bigcup_{j \in J} I_j$. Then the subset L weakly distinguishes the set M.

Proof. We choose two different elements $x, y \in M$.

a) At first we will suppose that both elements lye in the same $\mathcal{C}M_{j_0}$.

i) We will study the case $x \circ y \cap L = \emptyset$ then with respect to the condition of the construction of the covering there exists a dual atom d_{j_0} in $\mathcal{C}M_{j_0}$ such that $x \leq d_{j_0}$ and $y \parallel d_{j_0}$. Then it is sufficient to put $u = d_{j_0}$ and we have $u \circ x = d_{j_0} \circ x = d_{j_0} = d_{j_0} \notin L$ and $u \circ y = d_{j_0} \circ y = I_{j_0} \in L$.

ii) We entertain the second possibility, it is $x \circ y \cap L \neq \emptyset$. Then $x \circ y = I - j_0$. The elements x, y are other hence at least one of them is not equal to I_{J0}, for example x and we put $d = y$. Hence $d \circ y = x \circ y = I_{j_0}$ and $d \circ y = y \circ y = y \not{I}_{j_0}$.

b) Hereafter we will suppose that the elements x, y lye in other subhypergroupoids, in particular $x \in \mathcal{C}M_{j_1}, y \in \mathcal{C}M_{j_2}$. Let both of them lye in L, it is $x = I_{j_1}, y = I_{j_2}$. We subdivide this part of the proof into two fractions.

i) At first we will suppose that $x < y$. Then there exists an element $z \in M$ for which $x < z < y$ holds. Simultaneously from the construction of the covering of $\mathcal{CM}$ follows that $z \notin L$ We can put $u = z$ and we obtain $u \circ x = z \circ x = z \notin L$ and $u \circ y = z \circ y = y \in L$.

ii) The next possibility is that $x \parallel y$ (the elements x, y are incomparable). Now let $(x \circ y) \cap L = \emptyset$. Then it is sufficient to put either $u = x$ or $u = y$. More complete is the situation when $(x \circ y) \cap L \neq \emptyset$. We can choose a concrete element $I_{j_0} \in x \circ y$ and to this one we can find an element $z, z \prec I_{j_0}, z \notin L, x < z, z \parallel y$ for which $x \circ z = z \notin L$ and $I_{j_0} \in z \circ y$. We have found $u(= z)$ for which $u \circ x \notin L$ and $(u \circ y) \cap L \neq \emptyset$.

Now let exactly one of the elements x, y lyes in the subset L. For example x. Let $(x \circ y) \cap L \neq \emptyset$ Then it is sufficient to put $u = y$ and $u \circ x = y \circ x$ and $(y \circ x) \cap L \neq \emptyset$ simultaneously $u \circ y = y \circ y \notin L$. The second possibility is $(x \circ y) \cap L = \emptyset$. Now we put $u = x$ and we obtain $u \circ x = x \circ x = x \in L$ and $u \circ y = x \circ y$. Hence $(x \circ y) \cap L = \emptyset$.

Finally we suppose that no of the elements x, y lyes in L. Then if $(x \circ y) \cap L \neq \emptyset$ it is sufficient to put either $u = x$ or $u = y$. It remains the event $(x \circ y) \cap L = \emptyset$. From the construction of $\mathcal{CM}$ there follows the existence of $I_{j_1} \in \mathcal{C}M_{j_1}$ for which $x < I_{j_1}$ and $I_{j_2} \in \mathcal{C}M_{j_2}$ such that $y < I_{j_2}$. We obtain from the definition of the operation $\circ$ on hypergroupoid $\mathcal{M}$ that $I_{j_1} \circ I_{j_2} \subseteq I_{j_1} \circ y \subseteq x \circ y$ and similarly $I_{j_1} \circ I_{j_2} \subseteq x \circ I_{j_2} \subseteq x \circ y$. Hence we can put $u =$ either I_{j_1} or I_{j_2} and we obtain $u \circ x = I_{j_1} \circ x = I_{j_1} \in L$ and $u \circ y = I_{j_1} \circ y \subseteq x \circ y$ where $(x \circ y) \cap L = \emptyset$. The $u = I_{j_2}$ selection brings analogous results.

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