Asymptotic behavior of solutions of systems of difference equations with an application to delayed discrete equation

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In this contribution we investigate the asymptotic behavior for $k \to \infty$ of the solutions of the system of m difference equations

$$\Delta u(k) = F(k, u(k)) \tag{1}$$

where k is the independent variable assuming values from the set $N(a) := \{a, a+1, \dots\}$ with a fixed $a \in \mathbb{N}$, $u = (u_1, \dots, u_m)$, $\Delta u(k) = u(k+1) - u(k)$, and $F\colon N(a) \times \mathbb{R}^m \to \mathbb{R}^m$, $F = (f_1, \dots, f_m)$.

Our aim is to find sufficient conditions with respect to the right-hand side of system (1) which guarantee the existence of at least one solution $u(k) = (u_1^*(k), \dots, u_m^*(k))$, $k \in N(a)$ satisfying

$$(k, u_1^*(k), \dots, u_m^*(k)) \in \Omega(k),$$

where

$$\Omega(k) := \{(k, u) : k \in N(a),\ b_i(k) < u_i < c_i(k), i = 1, \dots, m\}$$

with $b_i, c_i : N(a) \to \mathbb{R}$, $i = 1, \dots, m$, being auxiliary functions such that $b_i(k) < c_i(k)$ for each $k \in N(a)$. (Such set Ω is called a polyfacial set.)

In paper [1] the above described problem is solved via Liapunov type technique. Here we combine this technique with the retract type technique which was used in paper [2].

To avoid defining too many new notions, we present the main result in a slightly different way than it is stated in the full version of this contribution.

Theorem 1 *Let $b_i(k)$, $c_i(k)$, $b_i(k) < c_i(k)$, $i = 1, \dots, m$, be real functions defined on $N(a)$ and let $f_i : N(a) \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$, be functions that are continuous with respect to all their arguments except the first one. Suppose that there exists a fixed index $j \in \{1, \dots, m\}$ for which the following condition holds:*
If $(k, u) \in \partial\Omega(k)$ is a point such that $u_j = b_j(k)$, then

$$f_j(k, u) < b_j(k+1) - b_j(k),$$

and if $(k, u) \in \partial\Omega(k)$ is a point such that $u_j = c_j(k)$, then

$$f_j(k, u) > c_j(k+1) - c_j(k).$$

Further suppose that for every $(k,u) \in \Omega(k)$ *and* $i = 1,\dots,m$, $i \neq j$,

$$b_i(k+1) < u_i + f_i(k,u) < c_i(k+1).$$

Then there exists a solution $u = (u_1^*(k),\dots,u_n^*(k))$ *of system* (1) *satisfying the inequalities*

$$b_i(k) < u_i^*(k) < c_i(k), \quad i = 1,\dots,m, \tag{2}$$

for every $k \in N(a)$.

The proof of this theorem is performed by a contradiction. We suppose that there exists no solution satisfying inequalities (2) for every $k \in N(a)$. Under this supposition we prove that there exists a continuous mapping (a retraction) of a closed interval onto its both endpoints which is, by known facts, impossible.

The previous result can be applied to prove that there exists a positive and bounded solution of the discrete delayed equation

$$\Delta u(k+n) = -p(k)u(k) \tag{3}$$

where $k \in N(a)$ is the independent variable and $n \in \mathbb{N}$, $n \geq 1$, is the fixed delay. The function $p : N(a) \to \mathbb{R}$ is supposed to be positive.

Theorem 2 *Let* $a \in \mathbb{N}$ *and* $n \in \mathbb{N}$ *be fixed. Suppose that there exists a constant* $\theta \in [0,1)$ *such that the function* $p : N(a) \to \mathbb{R}$ *satisfies the inequalities*

$$0 < p(k) \leq \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{1}{n+1} + \frac{\theta n}{8k^2}\right)$$

for every $k \in N(a)$. *Then there exists a solution* $u = u(k)$, $k \in N(a)$ *of equation* (3) *such that for* k *sufficiently large the inequalities*

$$0 < u(k) < \sqrt{k} \cdot \left(\frac{n}{n+1}\right)^k$$

hold.

To prove this theorem, we first rewrite the delayed equation (3) to a system of $n+1$ first order difference equations and then we show that this system satisfies all the assumptions of Theorem 1.

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References

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