On the two-scale approach to the heat transfer in buildings *

Stanislav Šťastník and Jiří Vala,
Brno University of Technology, Faculty of Civil Engineering
602 00 Brno, Veveří 95, Czech Republic
e-mail: Stastnik.S@fce.vutbr.cz, Vala.J@fce.vutbr.cz

Abstract

Understanding non-stationary thermal behaviour of buildings, its prediction and modification belong to priorities of the design of modern engineered buildings. Reliable mathematical models and software codes should contain information about a (quasi)periodic material microstructure; however, the macro- and microscales are allowed to differ dramatically, thus the standard finite element mesh refinement technique is not available. We shall combine the correction algorithm for the (not necessarily conforming) two-scale finite element discretization of special elliptic and parabolic problems (using the sequences of Rothe) with the two-scale homogenization. Using two levels of not necessarily nested grids, for our problem macroscopic convergence results similar to those known from the standard finite element approximation theory can be derived; more detailed local microstructural analysis can be done a posteriori.

1 Modelling of the heat transfer in buildings

Most materials used in civil engineering have a heterogenous structure, characterized by particles with quite different properties and pores (containing air, liquid water, etc.) of both micro- and macroscopic size. This brings difficulties to the reliable prediction of their thermal behaviour. The physical description of thermal transfer in buildings is based on the (seemingly simple) classical law of conservation of energy (and, alternatively, from further conservation laws – cf. [13], p. 38), formulated on a domain Ω in the 3-dimensional Euclidean space R^3; the mathematical equations of heat conduction (in a differential or an integral form) then incorporate material characteristics (for a stationary process at least a thermal conductivity, for a

*This paper has been supported by the project No. 1M6840770001 of the Ministry of Education, Youth and Sports of the Czech Republic, as a part of activities of the research centre CIDEAS, and the research project GA R 103/05/H044.

non-stationary one an additional characteristic derived from specific heat) whose reliable "effective values" (at some macroscopic level) are often not available (unlike those of separated particles) a priori. To complete the mathematical formulation of the problem, it is necessary to include some boundary (in the general non-stationary case also some initial) conditions. In most technical applications the Fourier boundary conditions of thermal convection from the outer environment (or between two neighbouring domains) are needed; other classes of boundary conditions (not discussed in this paper) are analyzed in [7], p. 207.

We shall use the standard notation of Sobolev spaces, compatible with [10]; moreover (for brevity) $V = W^{1,2}(\Omega)^3$, $H = L^2(\Omega)^3$ and $X = L^2(\partial\Omega)^3$. In general, we shall study the thermal behaviour of certain construction, occupying a domain Ω, in a time interval $\Theta = [0, T]$; T here is some prescribed positive time. The dot symbol will denote a time derivative everywhere. Following [12], our model problem is to find such temperature $\tau \in C(\Theta, H) \cap L^\infty(\Theta, V)$ with a time derivative $\dot{\tau} \in L^\infty(\Theta, H)$ that, in terms of scalar products $(.,.)$ in H or H^3, respectively, and $\langle .,. \rangle$ in X,

$$(v, \widetilde{C}\dot{\tau}) + (\nabla v, \widetilde{A}\nabla\tau) + \langle v, B\tau \rangle = \langle v, B\tau^\times \rangle \quad \forall\, v \in V \tag{1}$$

for certain given (time-variable) temperature $\tau^\times$ of outer environment in Θ and prescribed initial values τ_0 of τ on Ω in zero time. In this formulation A, B, C are positive material characteristics bounded on Ω or $\partial\Omega$: A is the thermal conductivity – for simplicity the same in all directions (also more complicated cases are frequently allowed to be transformed to this case, as discussed in [7], p. 206), C is the specific heat multiplied by the material density (both A and C are related to a domain Ω), B is the heat convection factor (related to the boundary $\partial\Omega$ of a domain Ω). The symbols $\widetilde{A}$, $\widetilde{C}$ refer to some "effective values" of A, C (rarely known a priori, unlike A and C); their construction is not trivial.

The evolution of $\tau^\times$ is rather slow (in practice quasiperiodic in day and year cycles); this argument seems to justify the idea to neglect the first additive term in (1) and prescribed τ_0 with the aim to obtain the more simple (stationary) problem: in any time $t \in \Theta$ to find such temperature $\tau(t)$ that

$$(\nabla v, \widetilde{A}\nabla\tau(t)) + \langle v, B\tau(t) \rangle = \langle v, B\tau^\times(t) \rangle \quad \forall\, v \in V\,. \tag{2}$$

This removes the effect of thermal accumulation, important in practice; nevertheless, we shall start with the analysis of (2) which can be considered as certain elliptic problem (in a fixed time t) and then we shall come to the more general parabolic problem (1).

2 Two-scale grids and two-scale homogenization

In both mathematical and engineering books and papers the definitions of and references to "two-scale problems" (or even "multiple-scale problems") occur in various senses. The very

rough classification of such approaches can be the following: i) some multiple levels of not necessarily nested grids are considered (and some successive corrections needed) without deeper analysis of microstructural phenomena, ii) the mathematical two-scale convergence theory is applied. The example of the original theoretical analysis of type i) is [4]. The fundamental definitions and lemmas of the theory ii) come from [11], [1] (for elliptic problems) and [5] (for parabolic problems); their extensive overview with remarks to technical applications (including the Dirichlet boundary problem for the heat equation) is contained in [3], much more references can be found in [14] and [16].

Let Λ be some subdomain of Ω; in practice $\operatorname{vol}\Omega \gg \operatorname{vol}\Lambda$. Let us assume that the decomposition of Ω into finite elements generates finite-dimensional subspaces V_h of V and the decomposition of Ω finite-dimensional subspaces V_δ of V; h and δ here are norms of such decompositions (upper bounds for characteristic lengths of their finite elements). It is easy to see (despite $h \to 0$) $h \gg \delta > 0$, thus the notation V_h and V_δ cannot be mismatched. We know that it is not realistic to assume some relation between V_h and V_δ a priori. Following [4], let us therefore introduce a new space (of higher finite dimension) $V_{h\delta} = V_h + V_\delta$. From the standard Friedrichs inequality (cf. [10], p. 216) and from the trace theorem, incorporating certain assumption on the regularity of the boundary (satisfied in reasonable technical applications – for details see [10], p. 222) we can easily derive that $a(.\,,.) = (\nabla.\,, \widetilde{A}\nabla.) + \langle .\,, B.\rangle$ defines a bilinear, symmetric, continuous and coercive form $a : V \times V \to R$. Since $a(.\,,.)$ can be identified with a scalar product in V and all above mentioned spaces are Hilbert ones, it is possible to make use of it to define operators of orthogonal projections $P_h : V_{h\delta} \to V_h$ and $P_\delta : V_{h\delta} \to V_\delta$; such operators will be useful for the design of the iterative algorithm, generating special sequences of approximate solutions of (1). The discretized forms of (1) for our two (macro- and micro-) scales are

$$(\nabla v_h, \widetilde{A}\nabla\tau_h) + \langle v_h, B\tau_h\rangle = \langle v_h, B\tau^\times\rangle \quad \forall\, v_h \in V_h\,, \tag{3}$$

$$(\nabla v_\delta, A(./\varepsilon)\nabla\tau_\delta^\varepsilon) + \langle v_\delta, B\tau_\delta^\varepsilon\rangle = \langle v_\delta, B\tau^\times\rangle \quad \forall\, v_\delta \in V_\delta\,; \tag{4}$$

we are seeking for $\tau_h \in V_h$ and for $\tau_\delta^\varepsilon \in V_\delta$ (in certain time t which is not emphasized here explicitly). The existence of solutions of (2), (3) and (4) follows from the Lax-Milgram theorem (cf. [3], p. 66).

It is not quite easy to construct $a(.\,,.)$ in practice because we do not know "homogenized thermal conductivity" $\widetilde{A}$ on Ω properly – at least on Λ (where we intend to analyze lower-scale phenomena) we have to obtain it by some homogenization process (via $\varepsilon \to 0$) from $a^\varepsilon(v, \tau^\varepsilon) = (v, A(./\varepsilon)\tau^\varepsilon)$ where some $A \in L^\infty(\Lambda \times R)$ is prescribed such that its values are Y-periodic in the second variable (we have $\widetilde{A}(x)$ and $A(x, y)$ with $x \in \Lambda$, $y \in Y$, the first variable x is not indicated in (3), (4)), Y is a unit cell in R^3 (a representative volume element of paralleliped shape, often rescaled as $Y = [0, 1]^3$) and $v, \tau^\varepsilon \in V$. To make the homogenization possible, we must require $a^\varepsilon(v, \tau^\varepsilon) \to a(v, \tau)$ (in R for each $v \in V$) as $\varepsilon \to 0$. The explicit formula for evaluation of $\widetilde{A}$ is known only for very special microstructures, namely for layered materials (cf. [3], p. 98, and [6]). However, the technique of the two-scale convergence (in more general

context, including strongly nonlinear problems and general measures, introduced in [16]) is available: since $\{\tau^\varepsilon\}$ (with positive decreasing ε) is a bounded sequence in V (at least in certain V^Λ, containing all restrictions of functions from V to Λ) then $\tau^\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \widehat{\tau}$ for certain $\widehat{\tau} \in L^2(\Lambda \times Y)$; the symbol $\rightharpoonup\!\!\!\!\rightharpoonup$ is reserved for the two-scale convergence in $L^2(\Lambda \times Y)$ in sense of [3], p. 176. Moreover, $\nabla\tau^\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \nabla_x\widehat{\tau} + \nabla_y\widehat{\tau}'$; here an additional function $\widehat{\tau}' \in L^2(\Lambda, W^{1,2}_{\#}(Y))$ (the index $_\#$ here forces the Y-periodicity) has the zero mean value $\int_Y \widehat{\tau}'(.,y)\,\mathrm{d}y = 0$ on Λ. Consequently also $\tau^\varepsilon \to \widehat{\tau}$ (where the second variable is omitted) in $L^2(\Lambda)$ and $A^\varepsilon(./\varepsilon)\nabla\tau^\varepsilon \rightharpoonup \widetilde{A}\nabla\widehat{\tau}$ (weakly) in $L^2(\Lambda)^3$. In general, setting a corresponding $\widetilde{A} \in L^2(\Lambda)$ (for a fixed $x \in \Lambda$ a constant) requires solving an auxiliary system of differential or integral equations (for details see [3], p. 112). Under more regularity assumptions such expensive calculations can be avoided (see [9] and other references from [14]).

3 Iterative computational algorithm for the stationary heat transfer

Let us consider an estimate τ^0 of a solution τ of (2). Let ω be certain real parameter, $0 < \omega < 2$. Following [4] (with slight modifications, coming from the two-scale analysis) and using the brief notation $b(v) = \langle v, B\tau^\times\rangle$ for any $v \in V$ let us suggest the following algorithm: i) find such $w^\varepsilon_\delta \in V_\delta$ that $a^\varepsilon(v_\delta, w^\varepsilon_\delta) = b(v_\delta) - a^\varepsilon(v_\delta, \tau^0)$ for all $v_\delta \in V_\delta$, ii) set $\tau^{\frac{1}{2}} = \tau^0 + \omega w^\varepsilon_\delta$, iii) find such $w_h \in V_h$ that $a(w_h, v_h) = b(v_h) - a(\tau^{\frac{1}{2}}, v_h)$ for all $v_h \in V_h$, iv) set $\tau^1 = \tau^{\frac{1}{2}} + \omega w_h$, v) replace τ^0 by τ^1, etc.

Analyzing this algorithm, using the projections P_h and P_δ, repeating step-by-step analogous calculations from [14], we obtain $w_h = P_h(\tau_{h\delta} - \tau^{\frac{1}{2}})$ and $w^\varepsilon_\delta = P_\delta(\tau_{h\delta} - \tau^0) + e_\delta$ where e_δ is a solution of an equation $a(v_\delta, e_\delta) = (a - a^\varepsilon)(v_\delta, w^\varepsilon_\delta - \tau^0)$ for all $v_\delta \in V_\delta$ and $\tau_{h\delta}$ comes from the discrete analogy of (2): find such $\tau_{h\delta} \in V_{h\delta}$ that

$$a(v_{h\delta}, \tau_{h\delta}) = b(v_{h\delta}) \quad \forall\, v_{h\delta} \in V_{h\delta}\,. \tag{5}$$

Consequently (as in [14] again) we have $\tau_{h\delta} - \tau^1 = (I - \omega P_h)(I - \omega P_\delta)(\tau_{h\delta} - \tau^0) - \omega(I - \omega P_h)e_\delta$ where I is an identity mapping. For simplicity, let us supply the Hilbert space V by the norm $\|.\| = \sqrt{a(.,.)}$; similar norms are admissible in finite-dimensional subspaces of V, too. By the strengthened Cauchy-Buniakowskiĭ-Schwarz inequality from [4] the norm of $(I - \omega P_h)(I - \omega P_\delta)$ is always (under the assumption $0 < \omega < 2$) lesser than 1; thus $\|\tau_{h\delta} - \tau^1\| \le \beta\|\tau_{h\delta} - \tau^0\| + \alpha\|e_\delta\|$ with some positive α and β where $\beta < 1$ and for an arbitrary integer n finally

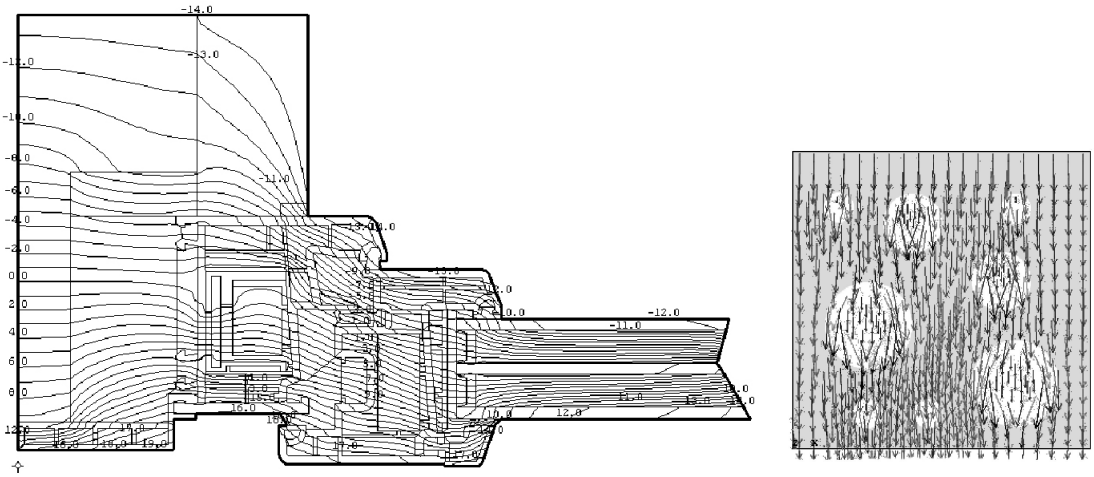
$$\|\tau_{h\delta} - \tau^n\| \le \beta^n\|\tau_{h\delta} - \tau^0\| + \frac{1 - \beta^n}{1 - \beta}\alpha\|e_\delta\|\,.$$

The limit passage $n \to \infty$ and $\varepsilon \to 0$ gives immediately $\|\tau_{h\delta} - \tau^n\| \to 0$. It remains to verify the expected limit relation between solutions $\tau_{h\delta}$ of (5) and τ of (2). Comparing (5) with (2)

with a test function $v_{h\delta} \in V_{h\delta}$ instead of $v \in V$, we have $a(v_{h\delta}, \tau - \tau_{h\delta}) = 0$. Assuming that $\tau \in V \cap W^{2,2}(\Omega)$ (although this requirement will be later removed), we can consider certain $\widetilde{\tau}$, defined in the following way: $\widetilde{\tau} = \tau$ on $\Omega \setminus \overline{\Lambda}$ and $\widetilde{\tau} = E\tau$ on Λ where E is a bounded extension operator from $W^{2,2}(\Omega \setminus \overline{\Lambda})$ to $W^{2,2}(\Omega)$. We are allowed to assume that the norm of $\widetilde{\tau}$ in $W^{2,2}(\Lambda)$ is not greater than the norm of τ in $W^{2,2}(\Lambda)$ and also that the norm of $\widetilde{\tau}$ in $W^{2,2}(\Omega)$ is not greater than the norm of τ in $W^{2,2}(\Omega \setminus \overline{\Lambda})$, multiplied by some generic constant; the validity of this assumption (due to the geometry of Ω and Λ) can be verified by means of the extension theorems in Sobolev spaces (for details see [2], pp. 264, 271 and 285).

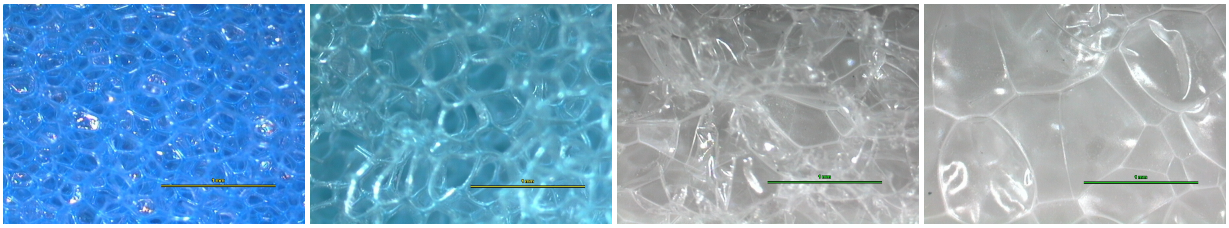
Let us now set $v_{h\delta} = \tau_{h\delta} - \widetilde{\tau}_{h\delta}$ where $\widetilde{\tau}_{h\delta} = r_h\widetilde{\tau} + r_\delta(\tau - \widetilde{\tau})$ and r_h and r_δ are the standard interpolants to the spaces V_h and V_δ, respectively. We obtain $a(v_{h\delta}, \tau - \widetilde{\tau}_{h\delta}) = a(v_{h\delta}, \tau_{h\delta} - \widetilde{\tau}_{h\delta}) = a(v_{h\delta}, v_{h\delta})$, from this the estimate $\|v_{h\delta}\|^2 \leq \|\tau - \tau_{h\delta}\|\|v_{h\delta}\|$, thus (if $\widetilde{\tau}_{h\delta} \neq \tau_{h\delta}$) $\|\tau_{h\delta} - \widetilde{\tau}_{h\delta}\| \leq \|\tau - \tau_{h\delta}\|$ and also $\|\tau - \tau_{h\delta}\| \leq \|\tau - \widetilde{\tau}_{h\delta}\| + \|\tau_{h\delta} - \widetilde{\tau}_{h\delta}\| \leq 2\|\tau - \widetilde{\tau}_{h\delta}\|$.

The last step is to verify $\|\tau - \widetilde{\tau}_{h\delta}\| \to 0$ as $h, \delta \to 0$. The classical results (rewritten in our notation) from the finite element interpolation theory (under some (semi)regularity assumption on the family of decomposition to finite elements) are $\|\widetilde{\tau} - r_h\widetilde{\tau}\| \leq \zeta h\|\tau\|_{W^{2,2}(\Omega\setminus\overline{\Lambda})}$, $\|(\tau - \widetilde{\tau}) - r_\delta(\tau - \widetilde{\tau})\| \leq \zeta\delta\|\tau - \widetilde{\tau}\|_{W^{2,2}(\Lambda)}$; ζ is a generic constant. The expected conclusion follows from this and from the estimate $\|\tau - \widetilde{\tau}_{h\delta}\| \leq \|\widetilde{\tau} - r_h\widetilde{\tau}\| + \|(\tau - \widetilde{\tau}) - r_\delta(\tau - \widetilde{\tau})\|$. In case of "insufficiently smooth" τ (τ may be not contained in $W^{2,2}(\Omega)$) the standard density argument ($V \cap W^{2,2}(\Omega)$ is dense in V) leads to the same conclusion.



Two illustrative figures show i) temperature isotherms in a part of certain advanced (macroscopic) window construction, ii) distributions of heat fluxes $-A\nabla\tau$ in a non-homogenous rubber-based insulation layer in great detail (a square with edge length 0.1 mm). Our original stationary problem in R^3 is reduced to the two-dimensional one, all computations are ANSYS-supported.

The following sequence of figures demonstrates some typical examples of microstructure of real insulation materials.



4 Generalization to the non-stationary heat transfer

The numerical analysis of the non-stationary heat transfer comes from the discrete forms of (1)

$$(v_h, \widetilde{C}(\tau_{sh} - \tau_{s-1\,h}))/\sigma + (\nabla v_h, \widetilde{A}\nabla\tau_{sh}) + \langle v_h, B\tau_{sh}\rangle = \langle v_h, B\tau_{sh}^{\times}\rangle \;\forall\, v_h \in V_h\,,$$

$$(v_\delta, C(./\varepsilon)(\tau_{s\delta}^{\varepsilon} - \tau_{s-1\,\delta}^{\varepsilon}))/\sigma + (\nabla v_\delta, A(./\varepsilon)\nabla\tau_{s\delta}^{\varepsilon}) + \langle v_\delta, B\tau_{s\delta}^{\varepsilon}\rangle = \langle v_\delta, B\tau_{s\delta}^{\times}\rangle \;\forall\, v_\delta \in V_\delta\,,$$

generalizing (3) and (4) in time $t = s\sigma$ (cf. (2)), $s \in \{1, \ldots, m\}$; we are seeking for $\tau_{sh} \in V_h$ and for $\tau_{s\delta}^{\varepsilon} \in V_\delta$, σ here is a real time interval length, $\sigma = T/m$ for some integer m where $m \to \infty$. The construction of $\widetilde{C}$ is even more simple than that of $\widetilde{A}$; for details see [3], p. 107, and [6]. Since τ_0 is known a priori, a parabolic problem can be substituted by a sequence of elliptic ones for particular s: instead of $a(.\,,.)$ we have some $(\nabla.\,, \widetilde{A}\nabla.) + (.\,, \widetilde{C}.)/\sigma$ and the right-hand side includes new additional terms $(v_h, \widetilde{C}\tau_{s-1\,h}))/\sigma$ or $(v_\delta, C(./\varepsilon)\tau_{s-1\,\delta}^{\varepsilon}))/\sigma$, respectively.

Using the technique of discretization in time, explained (even in more general context) in [15], we come to similar results as in the stationary case: applying the approach of [15], p. 592 (based on the construction of Rothe sequences of abstract functions $\Theta \to V$, linear or constant on every time interval $\{t \in \Theta : (s-1)\sigma < t \leq s\sigma\}$, and on the Eberlein-Shmul'yan and Arzelà-Ascoli theorems) with respect to [5], p. 334 (with slight modifications, forced by the two-scale finite element discretization on Ω and Λ), we are able to verify that the limit passage $\sigma, h, \delta, \varepsilon \to 0$ generates a solution of (1) with the expected properties $\tau \in C(\Theta, H) \cup L^\infty(\Theta, V)$, $\dot{\tau} \in L^\infty(\Theta, H)$; the local recognition of $\tau_{s\delta}^{\varepsilon}$ for a finite ε and $\sigma, h, \delta \to 0$ ($h \gg \delta$, $s \in \{1, \ldots, m\}$ for $m \to \infty$) is available again.

We can conclude that in case of linearized problems of heat transfer (where A, B, C are independent of τ) the two-scale finite element method, applying the iterative algorithm with "effective values" of material characteristics, whose basic idea comes from [4], gives numerical results similar to those from the classical finite element analysis. Nevertheless, in more complicated (nonlinear) cases the evaluation of such homogenized material characteristics may be

very expensive or not available at all, although the corresponding initial and boundary value problems are solvable (as in [16] and [15]). Further generalizations are needed in practice: the heat transfer should be coupled with the moisture transfer (cf. [13], p. 45, and [8]) and incorporate chemical reactions (as carbonation), mechanical deformation, etc. The development of effective numerical algorithms simulating such processes and phenomena seems to belong to significant research directions of computational mechanics in the near future.

References

[1] G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal., Vol. 23, No. 6 (1992), 1482-1512.

[2] V. I. Burenkov, Sobolev Spaces on Domains (Cardiff University & Peoples Friendship University of Russia, Cardiff & Moscow, 1997).

[3] D. Cioranescu and P. Donato, An Introduction to Homogenization (Oxford University Press, Oxford, 2000).

[4] R. Glowinski, J. He, A. Lozinski, J. Rappaz and J. Wagner, Finite element approximation of multi-scale elliptic problems using patches of elements, Num. Math., Vol. 101, No. 4 (2005), 663-688.

[5] A. Holmbom, Homogenization of parabolic equations – an alternative approach and some corrector-type results, Appl. Math. Vol. 47, No. 5 (1997), 321-343.

[6] M. Kamiński, Homogenization-based finite element analysis of unidirectional composites by classical and multiresolutional techniques. Computer Meth. Appl. Mech. & Eng., Vol. 194, No. 18-20 (2005), 2147-2174.

[7] J. Kuneš, Modelling of Thermal Processes (SNTL, Prague, 1989, in Czech).

[8] B. Lacabame, Homogenization of heat and mass conservation equations in porous media Monografiás del Seminario Matemático García de Galdeano, No. 27 (2003), 369-376.

[9] A.-M. Matache and Ch. Schwab, Two-scale finite element method for homogenization problems, Seminar of Applied Mathematics, Rep. No. 99-18 (ETH-Zentrum, Zürich, 2001).

[10] V. G. Maz'ya, Sobolev Spaces (Leningrad University Press, Leningrad (St. Petersburg) 1985, in Russian).

[11] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal., Vol. 20, No. 3 (1989), 608-623.

[12] S. Šťastník and J. Vala, Thermal-insulation property of advanced constructions of windows, On the two-scale finite element method for the heat transfer in buildings, Modelling 3, Abstr. (West-Bohemian University in Pilsen (Czech Republic), 2005), 69.

[13] J. Vala and S. Šťastník, On the modelling of heat propagation in buildings, Building Research J., Vol. 52, No. 1 (2004), 31-55.

[14] J. Vala, Two-scale finite element techniques in engineering mechanics, J. Mech. Eng., Vol. 57, No. 3 (2006), 127-140.

[15] J. Vala, The method of Rothe and two-scale convergence in nonlinear problems, Appl. Math., Vol. 48, No. 6 (2003), 587-606.

[16] J. Vala, Two-scale limits in some nonlinear problems of engineering mechanics, Math. Comput. Simul., Vol. 61, No. 4-6 (2003), 177-185.