Boundary value problem for Helmhotz equation and RKHPUM

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Abstract

Reproducing kernel hierarchical partition of unity method (RKHPUM) together with Smooth particle hydrodynamic method (SPHM), Element free Galerkin method (EFGM), Generalized finite element method (GFEM) belongs to meshless methods. These numerical methods have been developed and studied during last thirty years. It is characteristic for them that, in opposite of the FEM, they do not require any explicit mesh in the begining of computation. The fact that no mesh has to be generated we appreciate in the solution of 3D structural mechanics problems, when we deal with large deformations, when we solve problems with singularities or problems with moving boundaries.

We present the Reproducing kernel hierarchical partition of unity method in this article and we find the solution of the Neuman's and Dirichlet's boundary value problems for Helmholtz equation by means of this method.

1 Introduction

If engineers need to solve some practical boundary problem they probably reach for FEM. The first task when it comes to the realization of this method is to choose proper mesh. It can bring sometimes a lot of difficulties. For instance, in case when we are solving problems for large deformations, it is often necessary to overmesh the domain during the computation. Also creating structured meshes for three-dimensional FEM analysis of solids can be difficult and time consuming. Therefore, there is considerable interest in exploring methods of numerical analysis that avoid or greatly simplify this meshing task.

Meshless methods reproduce new approach to numerical solving of boundary value problems. The unknown solution of the given boundary value problem u is approximated by u^h over arbitrary spaced nodes (particles) $x_1, x_2, \ldots, x_N \in \Omega$. The approximation u^h has form

$$u^h(x) = \sum_{I=1}^{N} \Psi_I(x) u_I$$

where $\Psi_I(x)$ are shape functions and u_I are nodal parameters. This approximation seems to be the same as in FEM. But there are some differences between the FEM and the meshless methods. This differences are based on in the manner of construction of the shape functions and their properties. Namely, the functions Ψ_I depend only on points $x_1, x_2, \ldots, x_N$ and not on explicit mesh. Owing to this the approximation u^h in conjunction with collocation or Galerkin method provides a mesh-free computational formulation.

Also the RKHPUM is in principle connection semidiscrete Galerkin method and shape functions, that are constructed by means of moving least square method.

2 Neuman's boundary value problem

We consider the next boundary value problem

$$u''(x) + 16^2 u(x) = x, \ u'(0) = u'(1) = 0 \tag{1}$$

that has the analytical solution

$$u_{exact} = -\frac{1}{16^3}\sin 16x - \frac{1}{16^3}\frac{\cos 16 - 1}{\sin 16}\cos 16x + \frac{x}{16^2}.$$

We find the weak solution of the problem (1). It means we find a function $u \in W^{1,2}(0,1)$ such that

$$-\int_0^1 (u'v')\,\mathrm{d}x + 16\int_0^1 uv\,\mathrm{d}x = \int_0^1 xv\,\mathrm{d}x \ \forall v \in W^{1,2}(0,1). \tag{2}$$

The numerical solutions received by means of the FEM, when we take $n = 8, 11, 32, 64$ nodes and when we consider linear "hat" functions as the basis functions, are given in Figure 1.

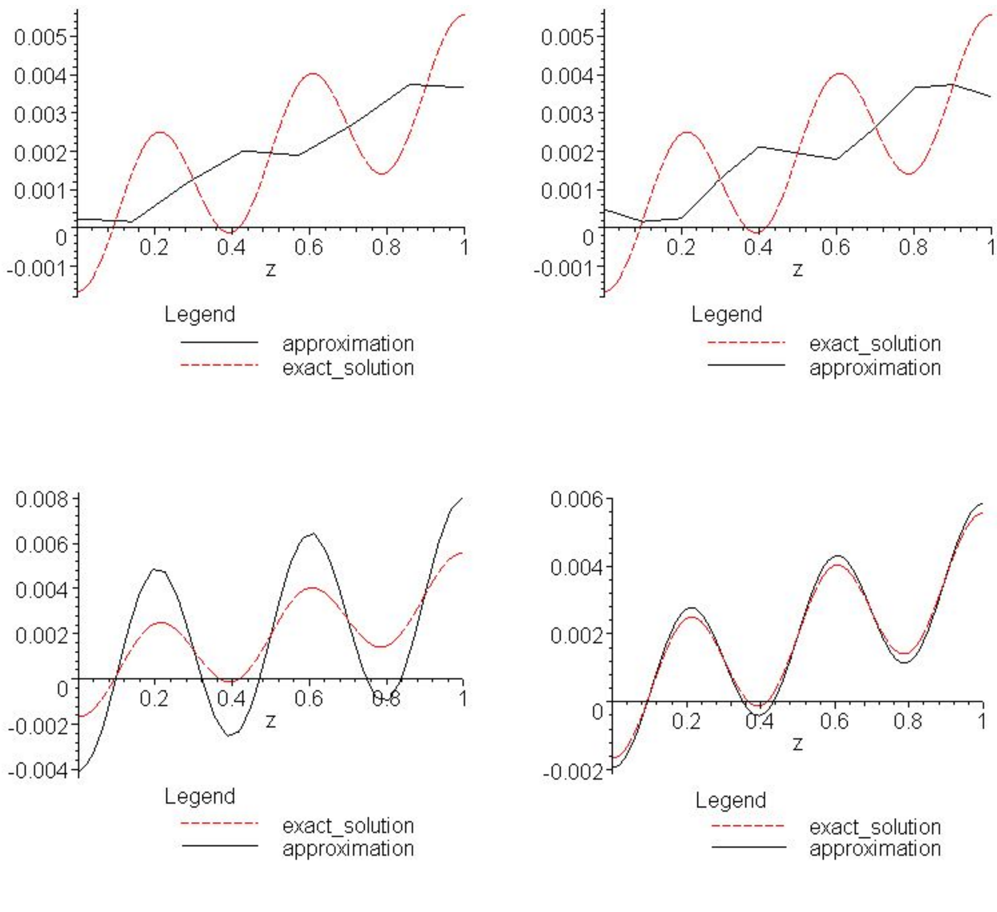

Figure 1: FEM - the approximation and the exact solution

Dependence the error $e = |u_{exact} - u_{FEM}|$ on the number of nodes is given in Table 1.

$n =$	8	11	32	64	91
$e \leq$	2.10^{-3}	22.10^{-4}	25.10^{-4}	3.10^{-4}	14.10^{-5}

Table 1: The error $e = |u_{exact} - u_{FEM}|$

We focus our attention on the RKHPUM now. We do our computation for N particles $x_1, \dots x_N$. Suppose the numerical solution in form

$$U(x) = \sum_{I=1}^{N} \Psi_I^0(x) U_I^0 + \sum_{I=2}^{N-1} \Psi_I^1(x) U_I^1, \tag{3}$$

where $\Psi_I^0(x),\ \Psi_I^1(x)$ are shape functions built for the first order polynomial basis $(1, x)$, the dilatation paremetr $R = 0.3$ and the weight function

$$\Phi(x) = \begin{cases} 1 - 2x^2 + x^4 & |x| \le 1 \\ 0 & \text{otherwise} \end{cases} .$$

It holds, that

$$\Psi_I^0(x) = p\left(\frac{x - x_I}{R}\right) b^0(x_I)\, \phi\left(\frac{x - x_I}{R}\right),$$

$$\Psi_I^1(x) = p\left(\frac{x - x_I}{R}\right) b^1(x_I)\, \phi\left(\frac{x - x_I}{R}\right)$$

The vectors $b^0,\ b^1$ are solutions of the linear systems

$$M(x) b^0 = (1, 0)^T,\ M(x) b^1(x) = (0, 1)^T$$

with

$$M(x, y) = \begin{pmatrix} m_0(x) & m_1(x) \\ m_1(x) & m_2(x) \end{pmatrix},$$

$$m_i(x) = \int_0^1 (y - x)^i \Phi\left(\frac{y - x}{R}\right) \mathrm{d}\, y,\ i = 0, 1, 2.$$

Substitution the (3) into the weak formulation (2) leads to the system of linear equations

$$AU = f,$$

where $U = (U_1^0, ..., U_{11}^0, U_2^1, ..., U_{10}^1)^T$,

$$f = (f_1^0, , .., f_{11}^0, f_2^1, ..., f_{10}^1)^T,\ f_I^\alpha = \int_0^1 f \Psi_I^\alpha \,\mathrm{d}x,\ \alpha \in \{0, 1\},$$

and if we denote $\Psi_{I,k}^\alpha = \frac{\partial \Psi_I}{\partial x_k}$,

$$A = \begin{pmatrix} A^{0,0} & A^{1,0} \\ A^{0,1} & A^{1,1} \end{pmatrix},\ A_{I,J}^{\alpha,\beta} = \int_0^1 (16^2 \Psi_I^\alpha \Psi_J^\beta - \Psi_{I,k}^\alpha \Psi_{J,l}^\beta) \,\mathrm{d}x,\ \alpha, \beta \in \{0, 1\}.$$

The exact solution and the solutions received by means of the RKHPUM for $N = 8, 11$ particles are given in Figure 2.
Dependence the error $e = |u_{exact} - u_{FEM}|$ on the number of particles is given in Table 2.

n=	8	10	11
$e \le$	7.10^{-4}	14.10^{-5}	8.10^{-5}

Table 2: Dependence the error $e = |u_{exact} - u_{RKHPUM}|$ on N.

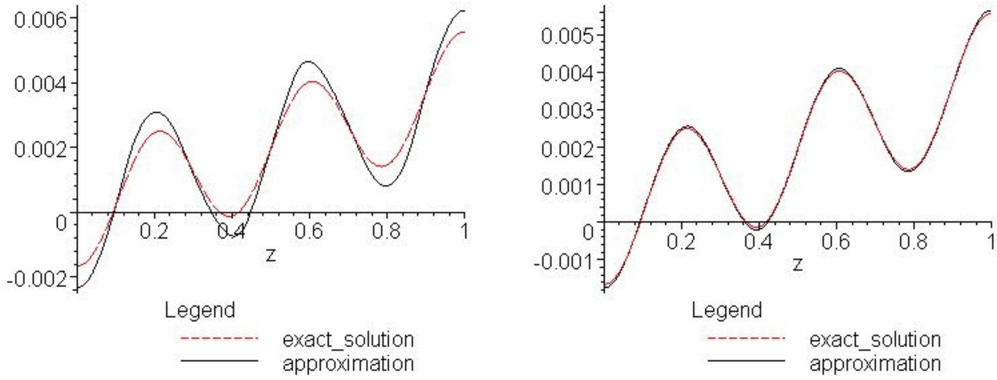

Figure 2: RKHPUM - the approximation and the exact solution

3 Dirichlet's boundary value problem

Consider the problem

$$u''(x) + 16^2 u(x) = x,\ u(0) = u(1) = 0. \tag{4}$$

It has classical solution

$$u_{exact} = \frac{1}{16^2} \frac{x \sin(16) - \sin(16\,x)}{\sin(16)}.$$

We solve the Dirichlet's boundary problem by means the RKHPUM now. We find a weak solution of the equation (4). It means we find a function $u \in W_0^{1,2}(0,1)$ such that

$$-\int_0^1 (u'v')\,\mathrm{d}x + 16^2 \int_0^1 uv\,\mathrm{d}x = \int_0^1 xv\,\mathrm{d}x\ \forall v \in W_0^{1,2}(0,1). \tag{5}$$

We suppose the same polynomial basis, the same dilatation parametr and the same weight function as in the previous section. Our approximation has the value described on the boundary. But approxinmative functions based on the moving least squares method do not exactly reproduce essential boundary conditions since they use base functions that are not strictly interpolants. It means, that there can be nodes x_J on boundary where $\Psi_I(x_J) \neq \delta_{IJ}$. Consequently,

$$u^h(x_J) = \sum_I \Psi_I(x_J) u_I \neq u_J.$$

One possibility, how to put this problem in order, is to work with the moment matrix

$$M(x,y) = \begin{pmatrix} m_0(x) & m_1(x) \\ m_1(x) & m_2(x) \end{pmatrix},$$

$$m_i(x) = \int_0^1 (x - 3x - x^2)^2 (\frac{y-x}{R})^i \Phi(\frac{y-x}{R})\,\mathrm{d}\,y,\ i = 0,1,2,$$

and new shape functions

$$\Psi_I^0(x) = p\left(\frac{x - x_I}{R}\right) b^0(x_I)(x - 3x - x^2)^2\, \phi\left(\frac{x - x_I}{R}\right),$$

$$\Psi_I^1(x) = p\left(\frac{x - x_I}{R}\right) b^1(x_I)(x - 3x - x^2)^2\, \phi\left(\frac{x - x_I}{R}\right)$$

The solution that we receive for $N = 6, 8$ particles is given in Figure 3.
Depencence the error $e = |u_{exact} - u_{RKHPUM}|$ on the number of nodes is given in Table 3.

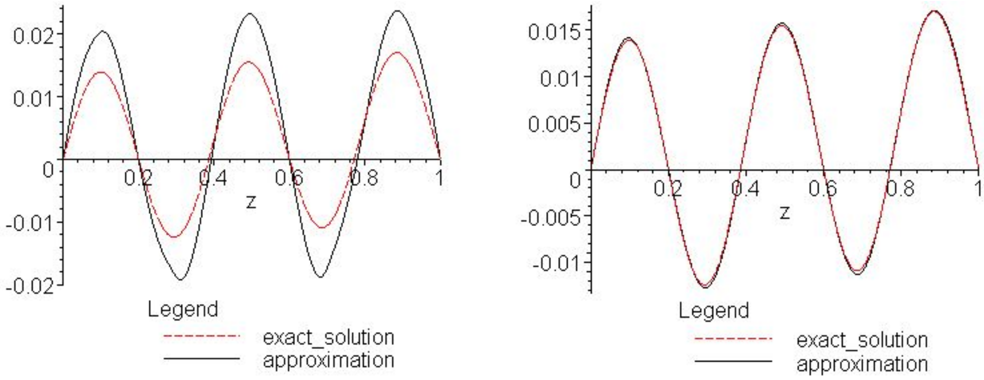

Figure 3: The RKHPUM approximation and the exact solution

n=	6	8	11
$e \leq$	8.10^{-5}	6.10^{-5}	2.10^{-5}

Table 3: Dependence the error $e = |u_{exact} - u_{RKHPUM}|$ on N.

4 Conclusion

We saw in given examples that the RKHPUM gives very good results. Received numerical solutions coresponds very well with character of the exact solutions. The errors are small for small number of particles. The problem is only with imposition of natural (Dirichlet) boundary condition.

References

[1] I. Babuška, U. Banerjee, J.E. Osborn, Survey of meshless and generalized finite element mehods: Anunified approach, Acta Numer. (2003) 1-125

[2] T. Belytschko, Y. Lu, I. Gu, Element-Free Galerkin Methods, Internat. J. Numer. Methods Engrg. 37 (1994) 229-256

[3] T. Belytschko, Y. Guo, W.K. Liu, A unified stability analysis of meshless particle methods, Internat. J. Numer. Methods Engrg. 48 (2000) 1359-1400,

[4] J.S. Chen, C. Pan, C.T. Wu, W.K. Liu, Reproducing kernel particle methods for large deformation analysis of non-linear structures, Comput. Methods Appl. Mech. Engrg. 139 (1996) 195-227

[5] J.S. Chen, C. Pan, C.T. Wu, Large deformation analysiss of rubber based on a reproducing kernel particle methods, Comput. Mech. 19 (1997) 211-227

[6] J.S. Chen, Ch.T. Wu, S. Yoon, Nonlinear version of stabilized conforming nodal integration for Galerkin meshfree methods, accepted by International Journal for Numerical Methods in Engeneering (2001)

[7] E. Cueto, M. Doblaré, L. Gracia, Imposing essential boundary conditions in the natural element method by means of density-scaled α-shapes. Internat. J. Numer. Methods Engrg. 49 (2000) 519-546

[8] W. Han, X. Meng, Some studies of the Reproducing kernel particle method, http://www.cs.uiova.edu/ftp/comp-math-rep/report-143.pdf

[9] P. Joyot, J. Trunzier, F. Chinesta, Enriched reproducing kernel approximation: Reproducing functions with discontinuous derivatives, in Meshfree methods for partial differential equation II, Springer, Berlin, 2004, pp 93 - 107

[10] Y. Krongauz, T. Belytschko, Enforcement of essential boundary conditions in meshless approximations using finite elements, http://tam.nwu.edu/kron /boundary.html (1995)

[11] S. Li, W.K. Liu, Reproducing kernel hierarchical partition of unity, Internat. J. Numer. Methods Engrg. 45 (1999) 251 - 317

[12] W.K. Liu, S. Jun, S. Li, J. Adee, T. Belytschko, Reproducing Kernel Particle Methods for Structural Dynamics, Internat. J. Numer. Methods Engrg. 38 (1999) 1655-1679

[13] W. Han, X. Meng, Error analysis of the reproducing kernel particle method, Comput. Methods Appl. Mech. Engrg. 190 (2001) 6157-6181

[14] J.J. Monaghan, Why Particle Methods Work. Sci. Stat. Comput. 3, No. 4 (1982) 422-433

[15] T. Strouboulis, I. Babuška, K. Copps, The design and analysis of the generalized finite element method, Comput. Methods Appl. Mech. Engrg. 181 (2000) 43-69