H_ν–STRUCTURE IS FIFTEEN

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Abstract

The aim of this paper is to remind the history of multistructures resp. hyperstructures mainly H_ν–structures which are celebrating the 15 years anniversary to be defined. The purpose is tho show how wide class H_ν–structures are.

1 Introduction

The first definition of hyperoperation and hypergroup was announcement by Frederic Marty in the 8^{th} Congress of Scandinavian Mathematicians in 1934. In this work exists a *motivation example* for this structure, which is the quotient of a group by any subgroup which is not an invariant subgroup. Marty used the *reproduction axiom* instead of the two axioms: the existence of neutral element and the existence of inverse element. He set the hypergroup free from the obligation to have neutral element and gave the possibility most widely used definition.

By the end of '80s the theory of hyperstructures had completed more than half of a century. At that time a lot of theory on hyperstructures had been achieved, for example, the relation β^* was studied in depth, the relations γ^* and ϵ^* were defined, special classes of hyperstructures as canonical hypergroups, feebly canonical, cogroups, very thin, P-hypergroups, hyper-lattices, hyperrings, hyperfields, and hyper-vector spaces were studied. Connections with other topics as geometry with join spaces, Steiner hypergroups, graphs and the topic of "fuzzy" were also near by.

In 1990, in Greece, T. Vougiouklis introduced the concept of the weak hyperstructures which are now named H_ν-structures. Over the last 15 years this class of hyperstructure, which is the largest, has been studied from several aspects as well as in connection with many other topics of mathematics.

2 BASIC DEFINITIONS

The hyperoperation $\star\colon H \times H \to \mathcal{P}^*(H)$ is called *weakly associative hyperoperation* (we abbreviate by WASS) if for any triad $a, b, c \in H$ $\big(a\star(b\star c)\big) \cap \big((a\star b)\star c\big) \neq \emptyset$. A *weak semihypergroup* (H_ν*-semigroup*) is a set H ($H \neq \emptyset$) equipped with a weakly associative hyperoperation. A H_ν-semigroup is called a *weak hypergroup* (H_ν*-group*) if moreover the reproduction axiom, i.e., $a \star H = H = H \star a$ is satisfied for any $a \in H$.

A *weak hyperring* (H_ν*-ring*) is a triad $(R, +, \cdot)$, where R is a set and $+\colon R \times R \to \mathcal{P}^*(R)$, $\cdot\colon R \times R \to \mathcal{P}^*(R)$ are weakly associative hyperoperations such that "+" satisfies the reproduction axiom $\big($i.e., $(R, +)$ is H_ν-group and $(R, \cdot)$ is a H_ν-semigroup$\big)$ and the hyperoperation "$\cdot$" is *weakly distributive* with respect to the hyperoperation "+", which means that for all elements $x, y, z \in R$ $x.(y + z) \cap (x.y + x.z) \neq \emptyset$, $(x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset$.

A H_ν-structure is called *very thin* iff all hyperoperations are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one.

3 SMALL SETS

The problem of enumeration and classification of H_ν-structures or of classes of them started from the beginning [4] but recently we have very interesting results mainly using computers. The problem becomes more complicated in H_ν-structures because we have very great numbers in this case. The partial order we introduced in the H_ν-structures [4] transfers and restrict the problem in finding the minimal, up to isomorphisms, H_ν-structures. In this direction we have results by Vougiouklis, Chung+Choi, but mainly and very recently by Lygeros in some papers by Bayon+Lygeros. Here there are some of their results: Let $H = \{a, b\}$ be a set of two elements, where we define a hyperoperation" $\cdot$", so we want to define a quadruple $(a \cdot a, a \cdot b, b \cdot a, b \cdot b)$. There are 20 H_ν-groups, up to isomorphism.

In the case of sets with three elements we have the following results:

Suppose we have a set $H = \{e, a, b\}$ where we define a hyperoperation "$\odot$" where there exists a scalar unit element e. The in order to define the H_ν-groups in H we want to define the quadruple $(a \cdot a, a \cdot b, b \cdot a, b \cdot b)$. Chung + Choi have proved that there are 13 minimal H_ν-groups. The number of all H_ν-groups with three elements, up to isomorphism, which have a scalar unit, is 292.

Some more general results, see [1]:

In a set with three elements there are, up to isomorphism, exactly 6494 minimal H_ν-groups. We remind that an H_ν-groups is called cyclic [4], if there is an element, called generator, which the powers have union the underline set. The minimal power with the above property is called period of the generator. Moreover if there exist an element and a special power, the minimum one, is the underline set, then the H_ν-groups is called single-power cyclic. After these definitions we specify that from the 6.494 non isomorphic H_ν-groups which are defined in a set with three elements the: 137 are abelians and the 6.357 are non-abelians; the 6.152 are cyclic and the 342 are not cyclic.

The total number of H_ν-groups with three elements, up to isomorphism, is 1.026.462. More precisely, there are 7.926 abelians and 1.018.536 non-abelians; the 1.013.598 are cyclic and the 12.864 are not cyclic. Finally the 16 of them are very thin.

The problem in the case of a set with four elements becomes, obviously, more complicate: The number of all H_ν-groups with four elements, up to isomorphism, which have a scalar unit, is 631.609. There are, up to isomorphism, 10.614.362 abelian hypergroups from which the 10.607.666 are cyclic and the 6.696 are not cyclic. There are, up to isomorphism, 8.028.299.905 abelian H_ν-groups from which the 7.995.884.377 are cyclic and the 32.415.523 are not cyclic.

All the data above are overtaken from the exciting lecture of T.Vougiuklis [4]. The H_ν-structures create realy a large class of hyperstrutures. Its a great fun to discover them.

References

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