H_{ν} -STRUCTURE IS FIFTEEN

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Abstract

The aim of this paper is to remind the history of multistructures resp. hyperstructures mainly H_{ν} -structures which are celebrating the 15 years anniversary to be defined. The purpose is tho show how wide class H_{ν} -structures are.

1 Introduction

The first definition of hyperoperation and hypergroup was announcement by Frederic Marty in the 8^{th} Congress of Scandinavian Mathematicians in 1934. In this work exists a *motivation example* for this structure, which is the quotient of a group by any subgroup which is not an invariant subgroup. Marty used the *reproduction axiom* instead of the two axioms: the existence of neutral element and the existence of inverse element. He set the hypergroup free from the obligation to have neutral element and gave the possibility most widely used definition.

By the end of '80s the theory of hyperstructures had completed more than half of a century. At that time a lot of theory on hyperstructures had been achieved, for example, the relation β^* was studied in depth, the relations γ^* and ϵ^* were defined, special classes of hyperstructures as canonical hypergroups, feebly canonical, cogroups, very thin, P-hypergroups, hyper-lattices, hyperrings, hyperfields, and hyper-vector spaces were studied. Connections with other topics as geometry with join spaces, Steiner hypergroups, graphs and the topic of "fuzzy" were also near by.

In 1990, in Greece, was organized by Thomas Vougiouklis a congress on hyperstructures (the algebraic structures equipped with at least one multivalued operation) which was first named AHA. It was the first Congress under the name AHA; however, it was 4th because there had already been three more congresses in Italy, organized by P. Corsini, on the same topic but under different names. During this congress T. Vougiouklis introduced the concept of the weak hyperstructures which are now named H_{ν} -structures. Over the last 15 years this class of hyperstructure, which is the largest, has been studied from several aspects as well as in connection with many other topics of mathematics.

2 Basic definitions

 H_{ν} -structures introduced in 1990 [11], is the hyperstructures satisfying the weak axioms where the non-empty intersection replaces the equality.

Some basic definitions:

Definice 1 The hyperoperation $\star : H \times H \to \mathcal{P}^*(H)$ is called weakly associative hyperoperation (we abbreviate by WASS) if for any triad $a, b, c \in H$

$$(a \star (b \star c)) \cap ((a \star b) \star c) \neq \emptyset.$$

The hyperoperation is weakly commutative (we abbreviate by COW) if for any $a, b \in H$

$$x \cdot y \cap y \cdot x \neq \emptyset.$$

Definice 2 A weak semihypergroup $(H_{\nu}\text{-semigroup})$ is a set H $(H \neq \emptyset)$ equipped with a weakly associative hyperoperation. A $H_{\nu}\text{-semigroup}$ is called a weak hypergroup $(H_{\nu}\text{-group})$ if moreover the reproduction axiom, i.e., $a \star H = H = H \star a$ is satisfied for any $a \in H$.

Morphisms considered in connection with these structures are so called H_{ν} -homomorphisms or a weak homomorphisms. Such a homomorphism is a mapping $f: H \to G$, where (H, \star) and (G, *) are H_{ν} -semigroups, such that for any pair $x, y \in H$

$$f(x \star y) \cap (f(x) \star f(y)) \neq \emptyset.$$

Definice 3 A weak hyperring $(H_{\nu}\text{-ring})$ is a triad $(R, +, \cdot)$, where R is a set and $+: R \times R \to \mathcal{P}^*(R)$, $:: R \times R \to \mathcal{P}^*(R)$ are weakly associative hyperoperations such that "+" satisfies the reproduction axiom (i.e., (R, +) is H_{ν} -group and (R, \cdot) is a H_{ν} -semigroup) and the hyperoperation " \cdot " is weakly distributive with respect to the hyperoperation "+", which means that for all elements $x, y, z \in R$

$$x.(y+z) \cap (x.y+x.z) \neq \emptyset, \quad (x+y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset.$$

Notice that unlike conventional rings the hyperoperation ``+" needn't be commutative.

Further, an H_{ν} -ring homomorphism or a weak homomorphism of H_{ν} -ring $(R, +, \cdot)$ into another one (S, \oplus, \odot) is a mapping $f: R \to S$ such that for any pair $x, y \in R$

$$f(x+y) \cap (f(x)+f(y)) \neq \emptyset, \qquad f(x \cdot y) \cap (f(x) \cdot f(y)) \neq \emptyset.$$

For more definitions, results and applications on Hv-structures, see books [5, 6, 13] and the papers in the references. An extreme class of hyperstructures is defined as follows [12]:

Definice 4 A H_{ν} -structure is called very thin iff all hyperoperations are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one.

The fundamental relations β^* , γ^* and ϵ^* are defined, in H_{ν} -groups, H_{ν} -rings and H_{ν} -vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [13]. The way to find the fundamental classes is given by analogous theorems to the following:

Theorem 1 Let (H, \cdot) be a H_{ν} -group and let us denote by U the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then the fundamental relation β^* is the transitive closure of the relation β .

Remark 1 The main point of the proof of this theorem is that the relation β guaranties the validity of the following: Take two elements x, y such that $\{x, y\} \in u \subset U$ and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the "hyperproducts" of the above β -classes are "products", then, they are fundamental classes. Analogous remarks for the relations γ , in H_{ν} -rings, and ϵ , in H_{ν} -vector spaces, are also applied.

An element is called *single* if its fundamental class is a singleton.

Definice 5 Let (H, \cdot) and (H, \odot) be two H_{ν} -semigroups defined on the same set H. Hyperoperation "·" is called smaller than " \odot ", and " \odot " greater than "·", iff there exists an automorphism $f \in \operatorname{Aut}(H, \odot)$ such that $x \cdot y \subset f(x \odot y)$, for all $x \in H$. Then we write $\cdot \leq \odot$ and we say that (H, \odot) contains the (H, \cdot) . If (H, \cdot) is a structure then it is called basic structure and the (H, \odot) is called H_b -structure.

Theorem 2 Greater hyperoperations of the ones which are WASS or COW, are also WASS and COW, respectively.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_{ν} -field one uses the fundamental relation γ^* : The H_{ν} -ring $(R, +, \cdot)$ is called H_{ν} -field if the quotient R/γ^* is a field,[11, 13].

Let us denote ω^* the kernel of the canonical map from $R \to R/\gamma^*$; then we give the following definition: We call *reproductive* H_{ν} -field any H_{ν} -field $(R, +, \cdot)$ for which the following additional axiom is valid $x(R-\omega^*) = (R-\omega^*)x = R-\omega^*$ for all $x \in R-\omega^*$. The definition of the H_{ν} -field introduced a new class of hyperstructures [15]:

Definice 6 The H_{ν} -semigroup (H, \cdot) is called h/v-group if the quotient H/β^* is a group.

The h/v-groups are a generalization of the H_{ν} -groups because in h/v-groups the reproductivity is not necessarily valid. However, sometimes a kind of reproductivity of classes is valid. That means that if H is partitioned into equivalence classes $\sigma(x)$, then $x\sigma(y) = \sigma(xy) = \sigma(x)y$, for all $x \in H$ [14]. This leads the quotient to be reproductive. In a similar way the h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces etc, are defined.

3 Small sets

The problem of enumeration and classification of H_{ν} -structures or of classes of them was started from the beginning [11, 13] but recently we have very interesting results mainly using computers and very interesting algorithms. Vougiouklis, Migliorato, De Salvo, Nordo et al worked on calculating classical hyperstructures of low order. However, the problem becomes more complicate in H_{ν} -structures because we have very great numbers in this case. The partial order we introduced in the H_{ν} -structures [13] transfers and restrict the problem in finding the minimal, up to isomorphisms, H_{ν} -structures. In this direction we have results by Vougiouklis, Chung+Choi, but mainly and very recently by Lygeros in some papers by Bayon+Lygeros.

Here there are some of their results:

Let $H = \{a, b\}$ be a set of two elements, where we define a hyperoperation " \cdot ", so we want to define a quadruple $(a \cdot a, a \cdot b, b \cdot a, b \cdot b)$. There are 20 H_{ν} -groups, up to isomorphism, which are the following ones:

(a, b, b, a), (H, b, b, a), (a, H, b, a), (a, b, H, a), (H, a, a, b),
(H, H, b, a), (H, b, H, a), (a, H, H, a), (b, H, H, a), (H, H, a, b),
(H, a, H, b), (a, H, H, b), (H, a, a, H), (H, b, a, H), (H, a, b, H),
(H, H, H, a), (H, H, H, b), (H, H, a, H), (H, H, b, H), (H, H, H, H).

In the case of sets with three elements we have the following results:

Suppose we have a set $H = \{e, a, b\}$ where we define a hyperoperation " \odot " where there exists a scalar unit element e. The in order to define the H_{ν} -groups in H we want to define the quadruple $(a \cdot a, a \cdot b, b \cdot a, b \cdot b)$. Chung + Choi have proved that there are 13 minimal H_{ν} -groups which are the following ones:

(b, e, e, a), (e,b, a, a, e), (e,a,b, a,b, e), (a, e,b, e,b, a),

(a,b, e,a, e,a, e), (H, e,b, a, e,a), (H, a, e,b, e,a), (a, H, H, e),

(b, H, H, e), (a, H, H, b), (H, b, a, H), (H, a, b, H), (H, a, b, e, H).

The number of all H_{ν} -groups with three elements, up to isomorphism, which have a scalar unit, is 292.

3.1 Some more general results

In a set with three elements there are, up to isomorphism, exactly 6494 minimal H_{ν} -groups. We remind that an H_{ν} -groups is called cyclic [13], if there is an element, called generator, which the powers have union the underline set. The minimal power with the above property is called period of the generator. Moreover if there exist an element and a special power, the minimum one, is the underline set, then the H_{ν} -groups is called single-power cyclic. After these definitions we specify that from the 6.494 non isomorphic H_{ν} -groups which are defined in a set with three elements the: 137 are abelians and the 6.357 are non-abelians; the 6.152 are cyclic and the 342 are not cyclic.

The total number of H_{ν} -groups with three elements, up to isomorphism, is 1.026.462. More precisely, there are 7.926 abelians and 1.018.536 non-abelians; the 1.013.598 are cyclic and the 12.864 are not cyclic. Finally the 16 of them are very thin.

The problem in the case of a set with four elements becomes, obviously, more complicate [3]:

The number of all H_{ν} -groups with four elements, up to isomorphism, which have a scalar unit, is 631.609.

There are, up to isomorphism, 10.614.362 abelian hypergroups from which the 10.607.666 are cyclic and the 6.696 are not cyclic.

There are, up to isomorphism, 8.028.299.905 abelian H_{ν} -groups from which the 7.995.884.377 are cyclic and the 32.415.523 are not cyclic. All results mentioned above can be found in [1].

Other interesting topics and methods in this field are:

Uniting elements	$Theta\ hyperoperations$
Adding elements	$Removing, \ absorbing, \ merging$
Representation	$Constructions \ in \ representation \ theory$
Hypergroupoid algebra	etc.

The used data are overtaken for the exciting lecture of Thomas Vouigiouklis [13]. The H_{ν} -structures create really a large class of hyperstructures. Its a great fun to discover them.

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