Quadrature of a parabola

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parabolic segment

a plane object bounded by the arch of parabola with end points A, B and chord AB (we call it base)

Fig. 1: segment of a parabola with the base AB

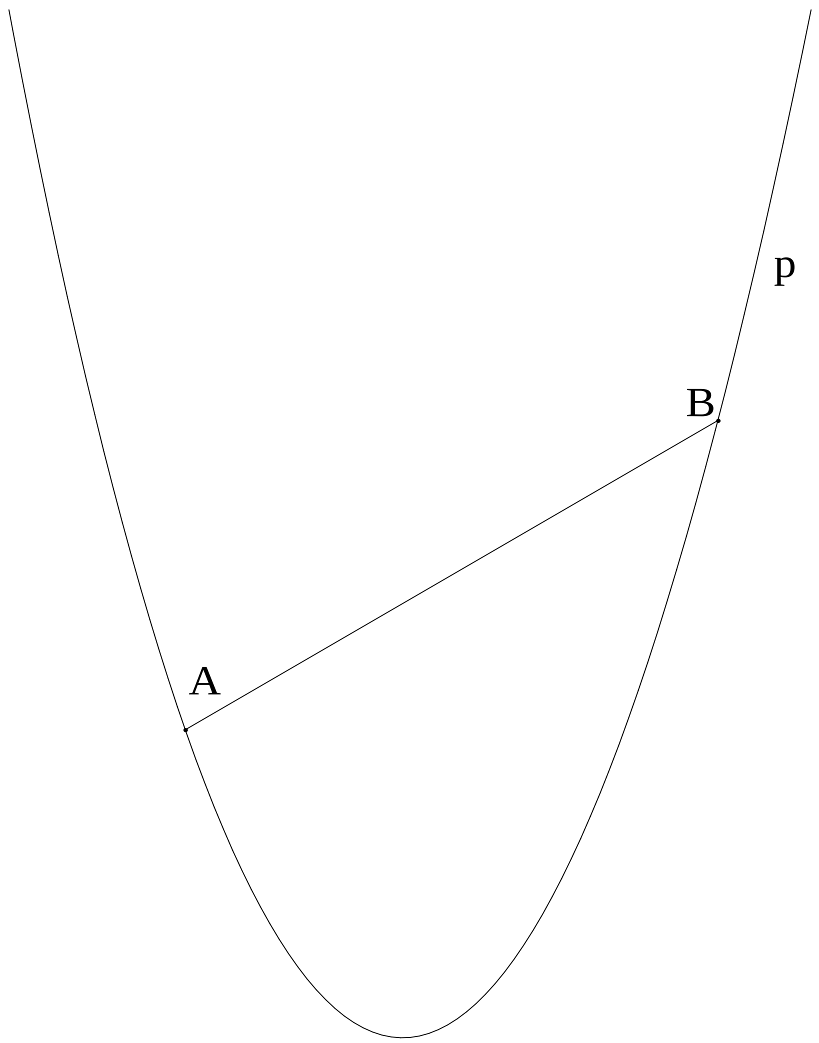

Fig. 1: segment of a parabola with the base AB

vertex of a parabola

a point where the tangent to the parabola, parallel to the base, is touching the parabola p

Fig. 2: vertex of a segment of parabola

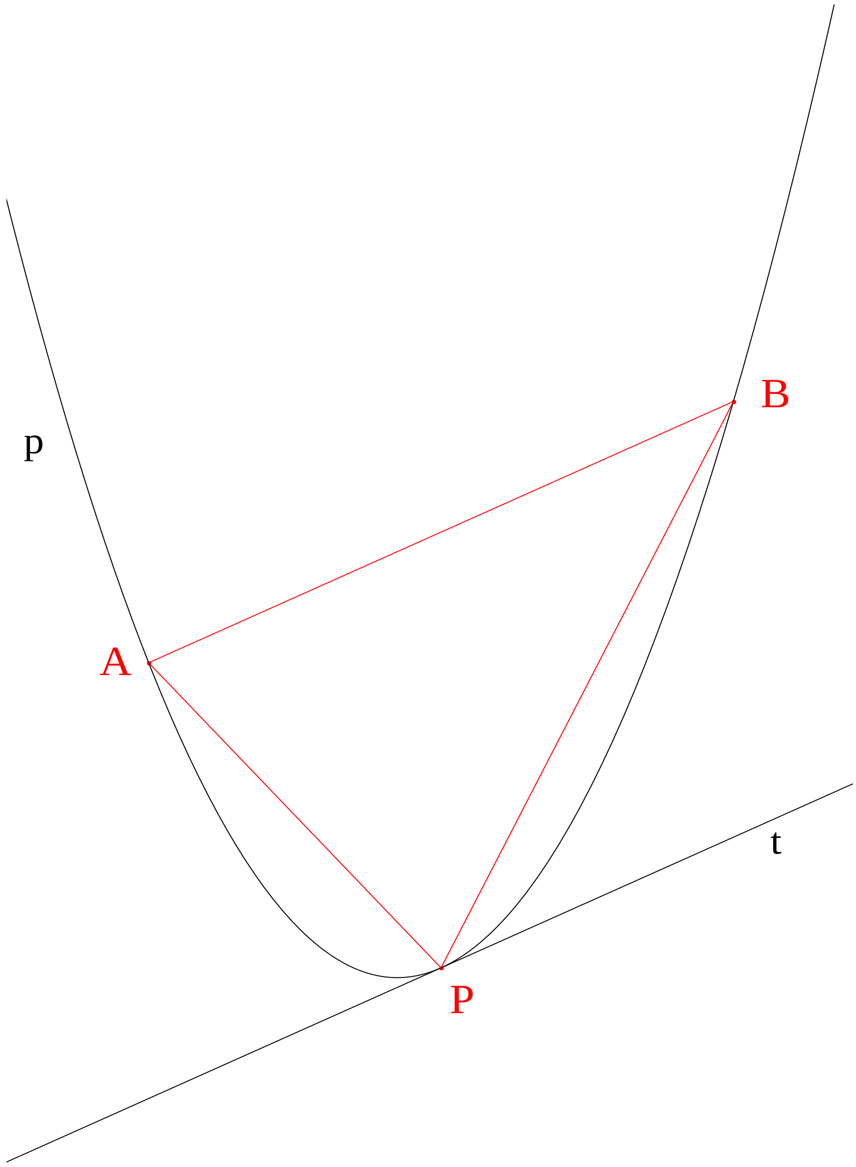

Fig. 2 : vertex of a segment of parabola

Archimedes' theorem:

The area of every segment of a parabola means four-thirds the area of a triangle with the same base AB and vertex P as the segment.

$$\mathbf{S} = \frac{\mathbf{4}}{\mathbf{3}}\mathbf{S}_{\triangle \mathbf{ABP}}.$$

Proposition 1

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as **PS*****,*** ***and if*** **AB** ***be a chord parallel to the tangent to the parabola at*** **P** ***and meeting*** **PS** ***in*** **S*****, then***

$$\mathbf{AS} = \mathbf{SB}.$$

Conversely, if $\mathbf{AS} = \mathbf{SB}$***, the chord*** **AB** ***will be parallel to the tangent at*** **P*****.***

Proposition 2

If in a parabola AB ***be a chord parallel to the tangent at*** P***, and if a straight line be drawn through*** P ***which is either itself the axis or parallel to the axis, and which meets*** AB ***in*** S ***and the tangent at*** A ***to the parabola in*** C***, then***

$$\mathbf{PS = PC}.$$

Fig. 3: focus definition of a parabola

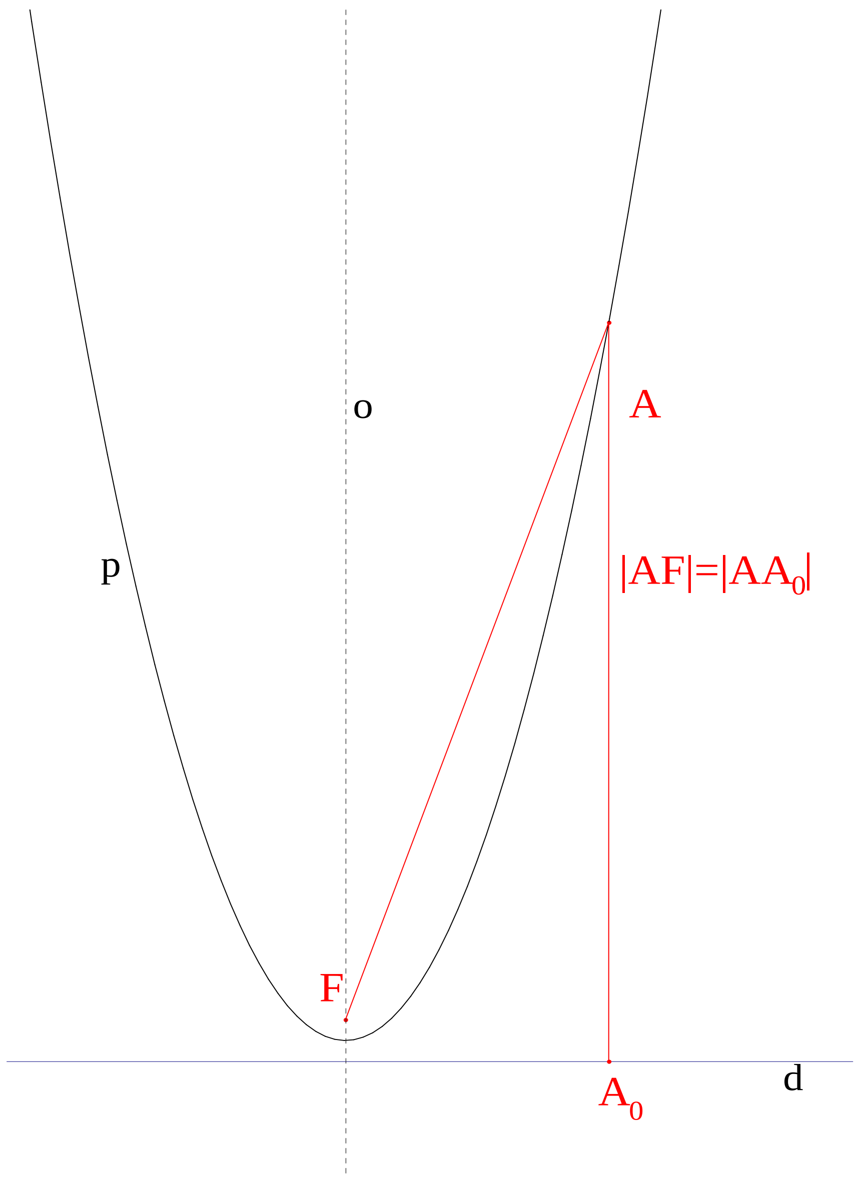

Fig. 3: focus definition of a parabola

Fig. 4 : tangent to a parabola

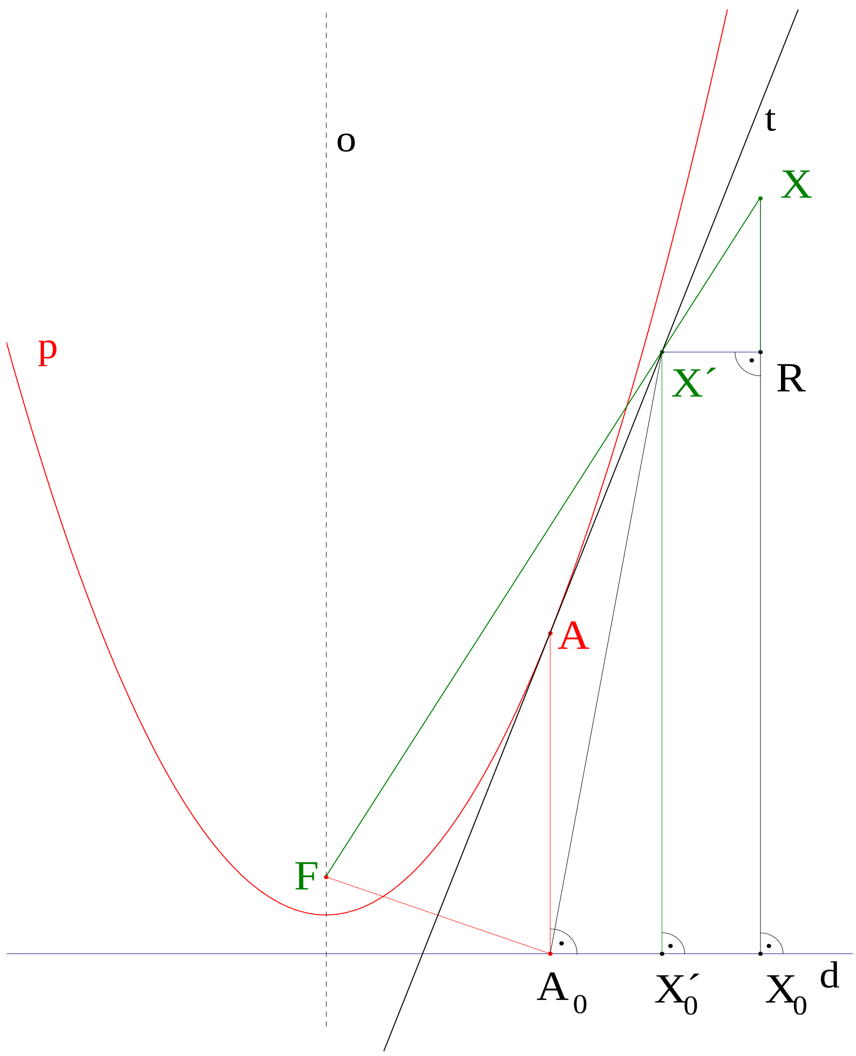

Fig. 4: tangent to a parabola

Archimedes' triangle:

Let C be the point of intersection of two tangents at different points A, B of parabola. The triangle ABC is Archimedes' triangle drawn to a segment of parabola with the base AB.

Fig. 5 : Archimedes' triangle drawn to a segment of parabola with the base AB

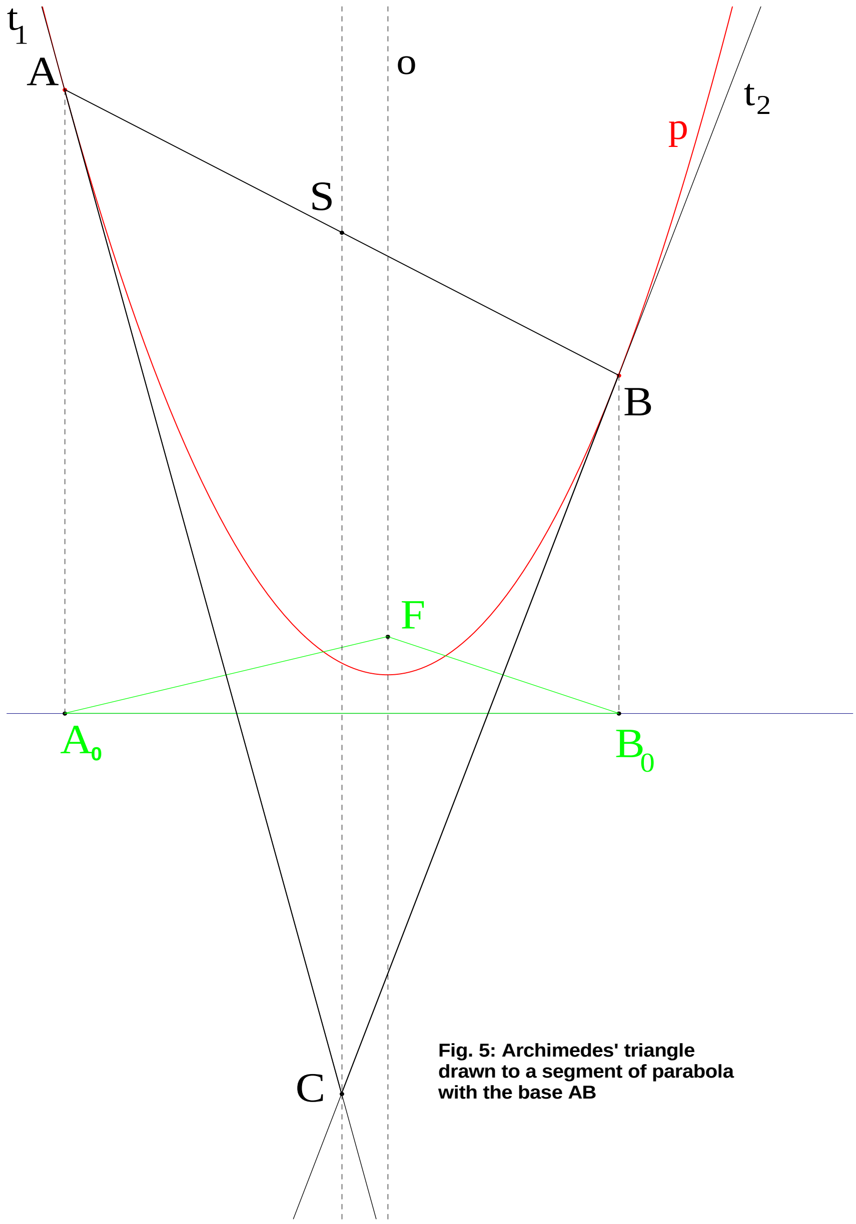

Fig. 5: Archimedes' triangle drawn to a segment of parabola with the base AB

The middle of the base AB of segment of parabola and vertex C of Archimedes' triangle lie on a line, which is parallel to the axis. (1)

In other words:

The median to the base of the Archimedes' triangle is parallel to the axis.

The vertex P of a segment of a parabola with the base AB is a middle of midline CS of Archimedes' triangle ABC. (2)

Division of the Archimedes' triangle:

The tangent A_1B_1 and chords AP, BP divide Archimedes' triangle ABC drawn to a segment of parabola with the base AB (of 1st level) into four triangles:

- **internal triangle APB bounded by the chords AB, AP and BP**
- **external triangle $\mathbf{A_1CB_1}$ bounded by the tangents at points A, B and P**
- **two residual triangles drawn to the segments with the bases AP and BP, which are also Archimedes' triangles (of 2nd level)**

Fig.6: division of the Archimedes' triangle

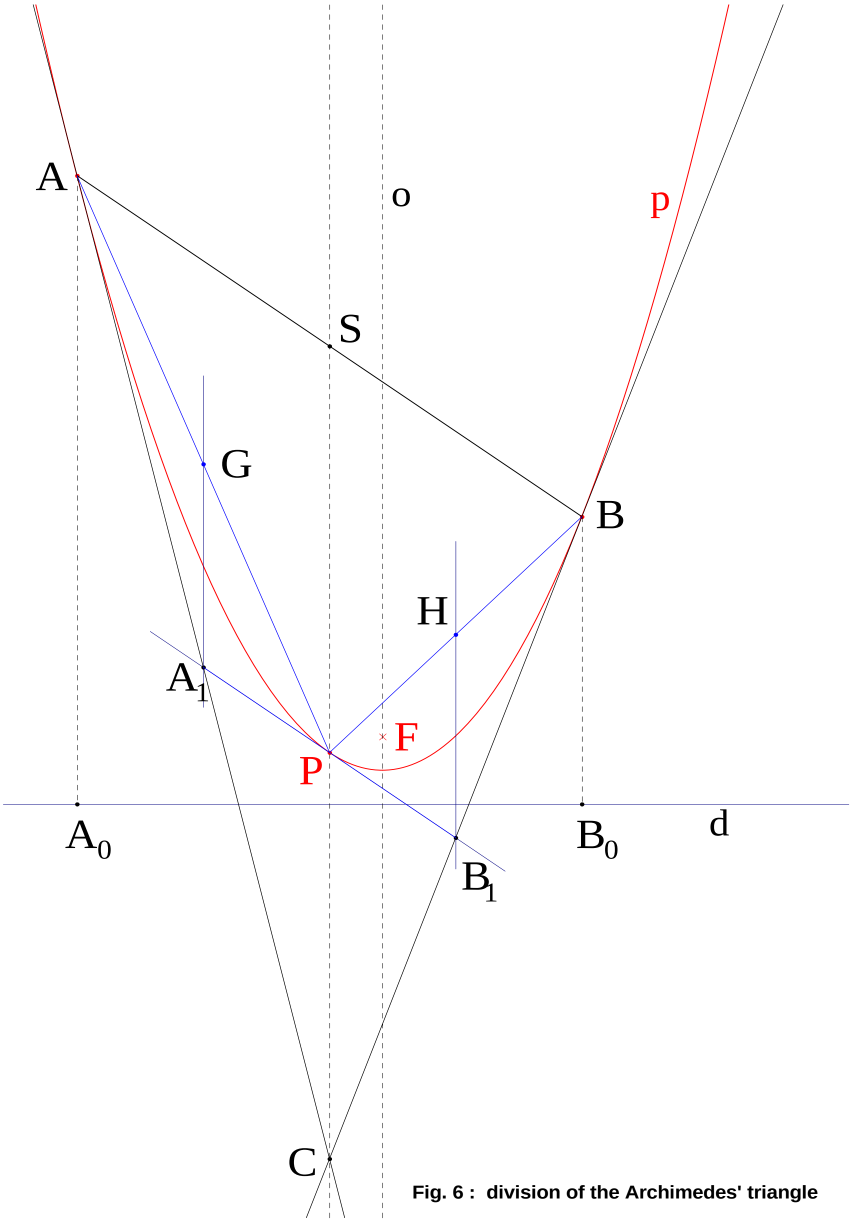

Fig. 6 : division of the Archimedes' triangle

Geometrical sequence for the areas of Archimedes' triangles:

$$\mathbf{S}, \frac{\mathbf{S}}{\mathbf{8}}, \frac{\mathbf{S}}{\mathbf{8^2}}, \dots$$

Geometrical sequence for the areas of the internal triangles:

$$\frac{\mathbf{S}}{\mathbf{2}}, \frac{\mathbf{S}}{\mathbf{2.8}}, \dots$$

Geometrical series of the areas of internal triangles (every area is multiplied by the number of triangles of each level):

$$\frac{\mathbf{S}}{\mathbf{2}} + \mathbf{2}.\frac{\mathbf{S}}{\mathbf{2.8}} + \mathbf{4}.\frac{\mathbf{S}}{\mathbf{2.8^2}} + \dots$$

The area of a parabolic segment:

$$\frac{\mathbf{S}}{\mathbf{2}} + \mathbf{2}.\frac{\mathbf{S}}{\mathbf{2.8}} + \mathbf{4}.\frac{\mathbf{S}}{\mathbf{2.8^2}} + \dots = \frac{\frac{\mathbf{S}}{\mathbf{2}}}{\mathbf{1} - \mathbf{2}.\frac{\mathbf{1}}{\mathbf{8}}} = \frac{\mathbf{2S}}{\mathbf{3}}$$

Reference

[1] The works of Archimedes, Edited by T.L.HEATH, Dover Publ. Inc., New York, 2002

[2] Heinrich Dörrie, 100 Great Problems of Elementary Mathematics, Their History and Solution, Dover Publ., New York, 1965

[3] Jindřich Bečvář, Ivan Štoll, Archimedes, největší vědec starověku, Prometheus, Praha, 2005