Qudrature of a parabola

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Problem of quadrature of some plane object is a problem of constructing a square of the same area as a given plane object using only a ruler and a pair of compasses.

Parabolic segment is a plane object, bounded by the arch of parabola with end points A, B and chord AB (we call it base). Vertex of parabola is a point where the tangent to the parabola, parallel with the base, is touching the parabola p.

Archimedes' theorem:

The area of every segment of parabola means four-thirds the area of a triangle with the same base AB and vertex P as the segment.

$$\mathbf{S} = \frac{\mathbf{4}}{\mathbf{3}} \mathbf{S}_{\triangle \mathbf{ABP}}.$$

Proposition 1

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PS, and if AB be a chord parallel to the tangent to the parabola at P and meeting PS in S, then

$$AS = SB.$$

Conversely, if $AS = SB$, the chord AB will be parallel to the tangent at P.

Proposition 2

If in a parabola AB be a chord parallel to the tangent at P, and if a straight line be drawn through P which is either itself the axis or parallel to the axis, and which meets AB in S and the tangent at A to the parabola in C, then

$$PS = PC.$$

Definition of Archimedes' triangle:

Let C be the point of intersection of two tangents at different points A, B of parabola. The triangle ABC is Archimedes' triangle drawn to a segment of parabola with the base AB.

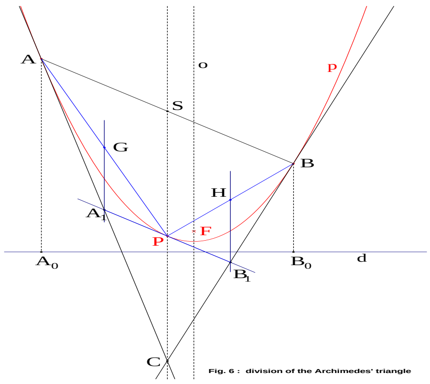

Fig. 6 : division of the Archimedes' triangle

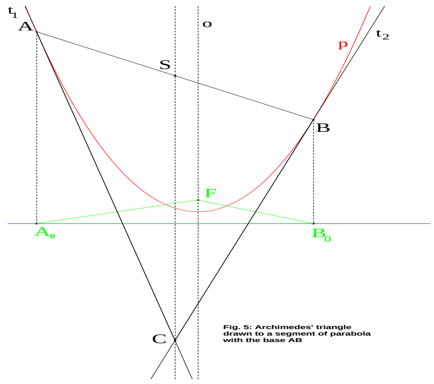

Fig. 5: Archimedes' triangle drawn to a segment of parabola with the base AB

Geometrical properties of the Archimedes' triangle:

The middle of the base AB of segment of parabola and vertex C of Archimedes' triangle lie on a line, which is parallel to the axis.

Another formulation:

The median to the base of the Archimedes' triangle is parallel to the axis.

The vertex P of a segment of a parabola with the base AB is a middle of midline CS of Archimedes' triangle ABC.

Division of the Archimedes' triangle:

- internal triangle APB bounded by the chords AB, AP and BP,
- external triangle A_1CB_1 bounded by the tangents at points A, B and P,
- two residual triangles drawn to the segments with the bases AP and BP, which are also Archimedes' triangles (of 2nd level)

If we mark the area of the Archimedes' triangle ABC as S, then the area of corresponding internal triangle is equal to $\frac{S}{2}$. In the same ratio 2:1 there are also the areas of internal and external triangle. Every residual triangle has an area of $\frac{S}{8}$.

We get geometrical sequence $S, \frac{S}{8}, \frac{S}{8^2}, \ldots$ for the areas of Archimedes' triangles.

We obtain geometrical sequence $\frac{S}{2}, \frac{S}{2.8}, \ldots$ for the areas of the internal triangles.

Geometrical series of the areas of the internal triangles:

$$\frac{S}{2} + 2.\frac{S}{2.8} + 4.\frac{S}{2.8^2} + \cdots = \frac{\frac{S}{2}}{1 - 2.\frac{1}{8}} = \frac{2S}{3}$$