Application of the Leray-Schauder Nonlinear Alternative

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Consider the following initial value problem

$$y'(t) = f(t, y(t), Ky(t)), \quad y(0) = 0 \quad a.e. \quad t \in I, \tag{1}$$

where $I = [0, T]$, $f : I \times R^n \times R^n \to R^n$, and $Ky(t) = \int_0^T \kappa(t, s, y(s))ds$ is a Fredholm operator with $\kappa : I \times I \times R^n \to R^n$.
Similarly problems have been investigated in [1,2,3]. Throughout this contribution the space of continuous functions on the interval I is denoted by C(I) with the norm $|.|_0$ given by

$$|y|_0 = \sup_{t\in[0,T]} |y(t)|.$$

$L^p(I)$, $1 \le p \le \infty$, is the space of measurable functions. For $y \in L^p(I)$the norm is given by

$$||y||_p = \left(\int_I |y(t)|^p dt\right)^{\frac{1}{p}} \ for \ 1 \le p < \infty,$$

$$||y||_p = ess \ \sup_{t\in I} |y(t)| \ for \ p = \infty.$$

Finally, AC(I) is the space of absolutely continuous functions on I.
To obtain an existence theorem for (1) there is used Leray-Schauder Nonlinear Alternative type:

Le C be a convex subset of a normed linear space E, and let U be an open subset of C with $p^* \in U$ *. Then every compact, continuous map* $\tilde{N} : \overline{U} \to C$ *has at least one of the following two properties:*

(I) $\tilde{N}$ *has a fixed point.*

(II) there is a $x \in \partial U$*, with* $x = (1 - \lambda)p^* + \lambda \tilde{N}x$ *for some* $0 < \lambda < 1$.

Definition 1. *A function* $f : I \times R^n \times R^n \to R^n$*, is* L^1 *-Carathéodory, if the following conditions hold:*

(i) the map $t \mapsto f(t, x, y)$ *is measurable for all* $(x, y) \in R^{2n}$*,*

(ii) the map $(x, y) \mapsto f(t, x, y)$ *is continuous for a.e.* $t \in I$*,*

(iii) for any $r > 0$ *, there exists* $h_r \in L^1(I)$*, such that,* $|f(t, x, y)| \le h_r(t)$*, for a.e.* $t \in I$*, and for all* $|x| \le r$*,* $|y| \le r$*.*

Definition 2: *A function $\kappa : I \times I \times R^n \mapsto R^n$ is integrable bounded L^1 - Carathéodory in t , if the following conditions hold:*

(i) *the map $s \mapsto \kappa_t(s, y) = \kappa(t, s, y)$ is measurable for all $y \in R^n$,*

(ii) *the map $y \mapsto \kappa_t(s, y)$ is continuous for a.e. $s \in I$,*

(iii) *for any $r > 0$, there exists $\mu r, t \in L^1(I)$, such that , $|y| \leq r$ implies $|\kappa_t(s, y)| \leq \mu_{r,t}(s)$ for a.e. $s \in I$,*

(iv) $\sup_{t\in I} \int_I \mu_{r,t}(s) ds < \infty$.

Theorem. *Suppose the following conditions are satisfied :*

(i) *the map $(t, s) \mapsto \kappa(t, s, x)$ is measurable for all $x \in R^n$, and the map $x \mapsto \kappa(t, s, y)$ is continuous for a.e. $(t, s) \in I \times I$,*

(ii) *κ is integrably bounded L^1 -Carathéodory in t,*

(iii) *f is L^1-Carathéodory .*

In addition, suppose there exists a constant $M > 0$, independent of λ , with $|y_0| < M$, $|y|_0 \neq M$ for any solution $y \in AC(I)$ of

$$y'(t) = \lambda f(t, y(t), Ky(t)), \quad y(0) = 0 \quad a.e.\ t \in [0, T],$$

for each $\lambda \in (0, 1)$. Then (1) has at least one solution $y \in AC(I)$.

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References

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