Three-dimensional systems of difference equations – the asymptotic behavior of their solutions

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In this contribution we study a system of three difference equations

$$\begin{aligned} \Delta u_1(k) &= f_1(k, u_1(k), u_2(k), u_3(k)), \\ \Delta u_2(k) &= f_2(k, u_1(k), u_2(k), u_3(k)), \\ \Delta u_3(k) &= f_3(k, u_1(k), u_2(k), u_3(k)), \end{aligned} \tag{1}$$

where $k \in N(a) := \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed, $\Delta u_i(k) = u_i(k+1) - u_i(k)$, $i = 1, 2$, and $f_1, f_2, f_3 \colon N(a) \times \mathbb{R}^3 \to \mathbb{R}$ are functions that are continuous with respect to their last three arguments, i.e. $f_i(k, u_1, u_2, u_3)$, $i = 1, 2, 3$, is continuous with respect to u_1, u_2 and u_3 for every fixed $k \in N(a)$.

The solution of system (1) is defined as an infinite sequence of number vectors

$$\{(u_1(k), u_2(k), u_3(k)\}_{k=a}^{\infty}$$

such that for any $k \in N(a)$ equalities (1) hold.

The existence and uniqueness of the solution of initial problem (1), (2) with

$$(u_1(a), u_2(a), u_3(a)) = (u_1^a, u_2^a, u_3^a) \in \mathbb{R}^3 \tag{2}$$

on $N(a)$ is obvious.

Our aim is to find sufficient conditions with respect to the right-hand side of system (1) which guarantee the existence of at least one solution

$$u^*(k) = (u_1^*(k), u_2^*(k), u_3^*(k)), \quad k \in N(a),$$

satisfying for every $k \in N(a)$

$$(k, u_1^*(k), u_2^*(k), u_3^*(k)) \in \Omega(k)$$

with

$$\Omega(k) := \{(k, u_1, u_2, u_3) : b_i(k) < u_i < c_i(k), i = 1, 2, 3\},$$

where $b_i, c_i : N(a) \to \mathbb{R}$, $i = 1, 2, 3$ are auxiliary functions such that $b_i(k) < c_i(k)$ for every $k \in N(a)$.

Obviously, each set $\Omega(k)$ is a rectangular parallelepiped and its boundary consists of six parts:

$$\partial\Omega(k) = \Omega_B^1(k) \cup \Omega_C^1(k) \cup \Omega_B^2(k) \cup \Omega_C^2(k) \cup \Omega_B^3(k) \cup \Omega_C^3(k)$$

with

$$\Omega_B^j(k) := \{(k, u_1, u_2, u_3) : k \in N(a), u_j = b_j(k), b_i(k) \le u_i \le c_i(k), i = 1, 2, 3, i \ne j\}$$

and

$$\Omega_C^j(k) := \{(k, u_1, u_2, u_3) : k \in N(a), u_j = c_j(k), b_i(k) \le u_i \le c_i(k), i = 1, 2, 3, i \ne j\}.$$

Theorem 1 *Let $b_i(k)$, $c_i(k)$, $b_i(k) < c_i(k)$, $i = 1, 2, 3$, be real functions defined on $N(a)$ and let $f_i : N(a) \times \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, be functions that are continuous with respect to their last three arguments. Suppose that for a fixed $j \in \{1, 2, 3\}$ and for every $k \in N(a)$ all the points of the sets $\Omega_B^j(k)$, $\Omega_C^j(k)$ are so called points of strict egress, i.e.,*

$$(k, u_1, u_2, u_3) \in \Omega_B^j(k) \quad \Rightarrow \quad f_j(k, u_1, u_2, u_3) < b_j(k+1) - b_j(k)$$

$$(k, u_1, u_2, u_3) \in \Omega_C^j(k) \quad \Rightarrow \quad f_j(k, u_1, u_2, u_3) > c_j(k+1) - c_j(k).$$

For the remaining indices $i \in \{1, 2, 3\} \setminus \{j\}$ suppose that for every $(k, u_1, u_2, u_3) \in \Omega(k)$, $k \in N(a)$,

$$b_i(k+1) < u_i + f_i(k, u_1, u_2, u_3) < c_i(k+1).$$

Then there exists a solution $u = (u_1^*(k), u_2^*(k), u_3^*(k))$ of system (1) *satisfying the inequalities*

$$b_i(k) < u_i^*(k) < c_i(k), \quad i = 1, 2, 3,$$

for every $k \in N(a)$.

The proof of Theorem 1 is performed by a contradiction and the so called retract type technique is used. The assumption that there exists no solution with the desired properties leads to the conclusion that there exists a continuous mapping (a retraction) of a closed interval onto its both endpoints which is, by known facts, impossible.

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