Join spaces of integral operators constructed from their group of free–member–combined type

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Abstract

This contribution is close to the investigation contained in [9, 10]. Linear integral operators belong to important tools in both classical pure and applied mathematics.

These topics are usually included into mathematical programmes on technical universities for the sake of their applicability in various engineering sciences.

The crucial idea is to investigate groups of linear integral operators on the same set of operators with different binary operations which are endowed with a suitable ordering of operators to obtain ordered groups of integral operators determined by Fredholm integral equations of the first and the second kind.

Using the standard functor of the transfer from the category of ordered groups and their isotone homomorphism into the category of hypergroups and their inclusion homomorphism we construct a hypergroup of integral operators or a hypergroup of classes of equivalence of hypergroups with suitable subhypergroups possessing interesting properties from the view of algebraic theory.

In this contribution we will construct a group of operators which can be also termed as an ordered group of integral operators of free–member–combined type.

1 Basic notions and definitions

Definition 1. *Hypergroupoid* is a pair $(H, *)$, where $H \neq \emptyset$ and $*\colon H \times H \to \mathbb{P}^*(H)$ (the system of all nonempty subsets of H) is a binary hyperoperation on H. If the associativity axiom $a * (b * c) = (a * b) * c$ holds for all $a, b, c \in H$ then the pair $(H, *)$ is called a *semihypergroup*. If moreover for any element $a \in H$ is satisfied the reproduction axiom $a * H = H = H * a$ then the pair $(H, *)$ is called a *hypergroup*.

Definition 2. A hypergroup $(H, *)$ is called a *transposition hypergroup* or a *join space* if it satisfies the transposition axiom: For all $a, b, c, d \in H$ the relation $b\backslash a \approx c/d$ implies $a * d \approx b * c$, (here $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$), where sets $b\backslash a = \{x \in H; a \in b * x\}, c/d = \{x \in H; c \in x * d\}$ are called *left and right extensions or fraction*, respectively.

The following lemma can be found in [5].

Lemma 1. *Let $(H, \cdot, \leq)$ be an ordered group. Define a hyperoperation $*\colon H \times H \to \mathbb{P}^*(H)$ by*

$$a * b = [a \cdot b)_{\leq} = \{x \in H; a \cdot b \leq x\}$$

for all pairs of elements $a, b \in H$. Then $(H, *)$ is a hypergroup which is commutative if and only if the group $(H, \cdot)$ is commutative.

An integral equation of the form

$$\varphi(x) - \lambda \int_a^b K(x,s)\varphi(s)\,\mathrm{d}s = f(x), \tag{1}$$

where $K(x,s)$ (kernel), $[x,s] \in \langle a,b\rangle \times \langle a,b\rangle \subset \mathbb{R}\times\mathbb{R}$, is a real or complex valued function (mostly real function), $f(x)$, $x \in \langle a,b\rangle \subset \mathbb{R}$, is a function called a free or an absolute member, λ is a numerical parameter and φ is an unknown function, is called Fredholm integral equation. More precisely, it is called *Fredholm integral equation of the second kind*, whereas an integral equation of the form

$$\int_a^b K(x,s)\varphi(s)\mathrm{d}s = f(x)$$

is called *Fredholm integral equation of the first kind.*

So we consider the operators

$$F(\lambda, K, f)\colon C\langle a,b\rangle \to C\langle a,b\rangle$$

($C\langle a,b\rangle$ means the set of all continuous functions on $\langle a,b\rangle$) of the type

$$F(\lambda,K,f)(\varphi(x)) = \lambda \int_a^b K(x,s)\varphi(s)\,\mathrm{d}s + f(x) \tag{2}$$

with a fixed interval $\langle a,b\rangle \subset \mathbb{R}$. The mentioned operator occurs in the construction of a series of functions which approximate the solution of Fredholm equation (1).

2 Construction of join space of operators based on ordered groups

In the sequel we will denote $C(J)$, $C(J\times J)$ the sets of continuous functions on $J = \langle a,b\rangle \subseteq \mathbb{R}$, $J\times J = \langle a,b\rangle \times \langle a,b\rangle \subseteq \mathbb{R}\times\mathbb{R}$, respectively, $f \in C(J)$, $K \in C(J\times J)$. Further we denote $C_+(J)$ the subset of all positive functions of $C(J)$.

For $J \subseteq \mathbb{R}$ let us denote

$$\mathbb{G} = \{F(\lambda,K,f) : K \in C(J\times J), f \in C(J), \lambda \neq 0\},$$

where $F(\lambda,K,f)$ is given by (2). For any pair of operators $F(\lambda_1,K_1,f_1), F(\lambda_2,K_2,f_2)$ in $\mathbb{G}$ let us define

$$F(\lambda_1,K_1,f_1) \odot F(\lambda_2,K_2,f_2) = F(\lambda_1\lambda_2, K_1K_2, \hat{K}_1 f_2 + f_1). \tag{3}$$

Here $\hat{K}_1(x,s) = K_1(x,x)$.

Proposition 1. *The groupoid $(\mathbb{G}, \odot)$ is a noncommutative group.*

Proof. Suppose $F(\lambda_i,K_i,f_i) \in \mathbb{G}$, $i = 1,2,3$. Then

$$\begin{aligned}
&\big(F(\lambda_1,K_1,f_1) \odot F(\lambda_2,K_2,f_2)\big) \odot F(\lambda_3,K_3,f_3) = \\
&= F(\lambda_1\lambda_2, K_1K_2, \hat{K}_1 f_2 + f_1) \odot F(\lambda_3,K_3,f_3) = F(\lambda_1\lambda_2\lambda_3, K_1K_2K_3, \hat{K}_1\hat{K}_2 f_3 + \hat{K}_1 f_2 + f_1) \\
&= F\big(\lambda_1\lambda_2\lambda_3, K_1K_2K_3, \hat{K}_1(\hat{K}_2 f_3 + f_2) + f_1\big) = F(\lambda_1,K_1,f_1) \odot (F(\lambda_2\lambda_3, K_2K_3, \hat{K}_2 f_3 + f_2) \\
&= F(\lambda_1,K_1,f_1) \odot \big(F(\lambda_2,K_2,f_2) \odot F(\lambda_3,K_3,f_3)\big),
\end{aligned}$$

thus the binary operation "$\odot$" is associative.
Further, for any operator $F(\lambda,K,f)\in\mathbb{G}$ and the operator $F(1,1,0)$ we have

$$F(\lambda,K,f)\odot F(1,1,0)=F(\lambda,K,f)=F(1,1,0)\odot F(\lambda,K,f),$$

thus the operator $F(1,1,0)$ is the unit of the semigroup $(\mathbb{G},\odot)$.
If $F(\lambda,K,f)\in\mathbb{G}$ is an arbitrary operator, then its inverse element within the monoid $(\mathbb{G},\odot)$ is the operator

$$F^{-1}(\lambda,K,f)=F\left(\frac{1}{\lambda},\frac{1}{K},-\frac{f}{\hat{K}}\right)=F\left(\lambda^{-1},\frac{1}{K},-\frac{f}{\hat{K}}\right).$$

Indeed,

$$F(\lambda,K,f)\odot F\left(\lambda^{-1},\frac{1}{K},-\frac{f}{\hat{K}}\right)=F(1,1,0)=F\left(\lambda^{-1},\frac{1}{K},-\frac{f}{\hat{K}}\right)\odot F(\lambda,K,f). \qquad \square$$

For $J\subseteq\mathbb{R}$ let us denote

$$\mathbb{G}_1=\{F(\lambda,K,f):K\in C_+(J\times J),K\neq 0,f\in C(J),\lambda\neq 0\}.$$

Evidently, $(\mathbb{G}_1,\odot)$ is subgroup of $(\mathbb{G},\odot)$. For any pair of operators $F(\lambda_1,K_1,f_1),F(\lambda_2,K_2,f_2)$ in $\mathbb{G}_1$ we put

$$F(\lambda_1,K_1,f_1)\leq F(\lambda_2,K_2,f_2) \text{ if and only if } \lambda_1=\lambda_2, K_1(x,s)=K_2(x,s) \text{ and } f_1(x)\leq f_2(x)$$

for any $[x,s]\in J\times J$.

Proposition 2. *The structure* $(\mathbb{G}_1,\odot,\leq)$ *is a noncommutative ordered group.*

Proof. From the definition of the relation "$\leq$" it follows immediately, that this relation is on $\mathbb{G}_1$ reflexive, antisymmetrical and transitive, hence the pair $(\mathbb{G}_1,\leq)$ is an ordered set.

It remains to verify the compatibility of the ordering "$\leq$" on $\mathbb{G}_1$ with the binary operation "$\odot$". Suppose $F(\lambda_1,K_1,f_1),F(\lambda_2,K_2,f_2)\in\mathbb{G}_1$ are integral operators satisfying $F(\lambda_1,K_1,f_1)\leq F(\lambda_2,K_2,f_2)$ and $F(\lambda,K,f)\in\mathbb{G}_1$ is an arbitrary operator. Then

$$f_1(x)\leq f_2(x), \qquad 0\neq\lambda_1=\lambda_2, \qquad K_1(x,s)=K_2(x,s)$$

for any $[x,s]\in J\times J$, which implies

$$\lambda\lambda_1=\lambda\lambda_2, \qquad KK_1=KK_2,$$
$$\hat{K}f_1+f\leq\hat{K}f_2+f$$

for each $[x,s]\in J\times J$, hence

$$F(\lambda,K,f)\odot F(\lambda_1,K_1,f_1)=F(\lambda\lambda_1,KK_1,\hat{K}f_1+f)\leq$$
$$\leq F(\lambda\lambda_2,KK_2,\hat{K}f_2+f)=F(\lambda,K,f)\odot F(\lambda_2,K_2,f_2).$$

Similarly,

$$\lambda_1\lambda=\lambda_2\lambda, \qquad K_1K=K_2K,$$
$$\hat{K}_1f+f_1\leq\hat{K}_2f+f_2$$

for each $[x,s]\in J\times J$, hence

$$F(\lambda_1,K_1,f_1)\odot F(\lambda,K,f)=F(\lambda_1\lambda,K_1K,\hat{K}_1f+f_1)\leq$$
$$\leq F(\lambda_2\lambda,K_2K,\hat{K}_2f+f_2)=F(\lambda_2,K_2,f_2)\odot F(\lambda,K,f).$$

Consequently, $(\mathbb{G}_1,\odot,\leq)$ is a noncommutative ordered group. $\square$

Now we apply the simple construction of a hypergroup from Lemma 1 to the considered ordered group of integral operators.

Thus for an arbitrary pair of operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{G}_1$ we define a hyperoperation $*\colon \mathbb{G}_1 \times \mathbb{G}_1 \to \mathbb{P}^*(\mathbb{G}_1)$ as follows:

$$\begin{aligned} F(\lambda_1, K_1, f_1) * F(\lambda_2, K_2, f_2) = & \qquad (4)\\ &= \{F(\lambda, K, f) \in \mathbb{G}_1 : F(\lambda_1, K_1, f_1) \odot F(\lambda_2, K_2, f_2) \le F(\lambda, K, f)\} \\ &= \{F(\lambda, K, f) \in \mathbb{G}_1 : F(\lambda_1\lambda_2, K_1K_2, \hat{K}_1 f_2 + f_1) \le F(\lambda, K, f)\} \\ &= \{F(\lambda_1\lambda_2, K_1K_2, f) : \hat{K}_1 f_2 + f_1 \le f\}. \end{aligned}$$

Then we obtain from Proposition 1 with respect to Lemma 1:

Proposition 3. *Let* $J = \langle a, b\rangle \subseteq \mathbb{R}$ *and* $*\colon \mathbb{G}_1 \times \mathbb{G}_1 \to \mathbb{P}^*(\mathbb{G}_1)$ *be the above defined binary hyperoperation. Then the hypergroupoid* $(\mathbb{G}_1, *)$ *is a noncommutative hypergroup.*

Now we are going to verify that the above constructed noncommutative hypergroup $(\mathbb{G}_1, *)$ is in fact a join space. The following auxiliary assertion will be very useful in the sequal.

Lemma 2. *Let* $J \subseteq \mathbb{R}$ *be a compact interval and* $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{G}_1$ *be arbitrary operators, i.e. elements of the hypergroup* $(\mathbb{G}_1, *)$*. Then*

1° $F(\lambda_1, K_1, f_1)/F(\lambda_2, K_2, f_2) = \Big\{F\Big(\dfrac{\lambda_1}{\lambda_2}, \dfrac{K_1}{K_2}, f\Big) : f \le f_1 - \dfrac{\hat{K}_1}{\hat{K}_2} f_2\Big\},$

2° $F(\lambda_2, K_2, f_2)\backslash F(\lambda_1, K_1, f_1) = \Big\{F\Big(\dfrac{\lambda_1}{\lambda_2}, \dfrac{K_1}{K_2}, f\Big) : f \le \dfrac{f_1 - f_2}{\hat{K}_2}\Big\}.$

Proof. Taking into account the fact that the function K_2 is positive and $\lambda_2 \ne 0$ on the whole $J \times J$ then with respect to the definitions of corresponding hyperoperations we obtain for arbitrary pairs of operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{G}_1$ that

$$\begin{aligned} F(\lambda_1, K_1, f_1)/F(\lambda_2, K_2, f_2) &= \{F(\lambda, K, f) : F(\lambda_1, K_1, f_1) \in F(\lambda, K, f) * F(\lambda_2, K_2, f_2)\} \\ &= \{F(\lambda, K, f) : F(\lambda_1, K_1, f_1) \ge F(\lambda, K, f) \odot F(\lambda_2, K_2, f_2)\} \\ &= \{F(\lambda, K, f) : F(\lambda_1, K_1, f_1) \ge F(\lambda\lambda_2, KK_2, \hat{K} f_2 + f)\} \\ &= \Big\{F\Big(\frac{\lambda_1}{\lambda_2}, \frac{K_1}{K_2}, f\Big) : f \le f_1 - \frac{\hat{K}_1}{\hat{K}_2} f_2\Big\}, \end{aligned}$$

which proves formula 1°.
Further

$$\begin{aligned} F(\lambda_2, K_2, f_2)\backslash F(\lambda_1, K_1, f_1) &= \{F(\lambda, K, f) : F(\lambda_1, K_1, f_1) \in F(\lambda_2, K_2, f_2) * F(\lambda, K, f)\} = \\ &= \{F(\lambda, K, f) : F(\lambda_1, K_1, f_1) \ge F(\lambda_2, K_2, f_2) \odot F(\lambda, K, f)\} \\ &= \{F(\lambda, K, f) : F(\lambda_1, K_1, f_1) \ge F(\lambda_2\lambda, K_2K, \hat{K}_2 f + f_2)\} \\ &= \Big\{F\Big(\frac{\lambda_1}{\lambda_2}, \frac{K_1}{K_2}, f\Big) : f \le \frac{f_1 - f_2}{\hat{K}_2}\Big\} \end{aligned}$$

and formula 2° is proved, as well. □

Theorem 1. $(\mathbb{G}_1, *)$ *is a noncommutative transposition hypergroup, i.e. a noncommutative join space.*

Proof. By Proposition 3 the hypergroupoid $(\mathbb{G}_1, *)$ is a noncommutative hypergroup. It remains to prove that this hypergroup satisfies the transposition law.

Suppose $F(\lambda_i, K_i, f_i) \in \mathbb{G}_1$, $i = 1, 2, 3, 4$ is a quadruple of integral operators such that

$$F(\lambda_2, K_2, f_2)\backslash F(\lambda_1, K_1, f_1) \approx F(\lambda_3, K_3, f_3)/F(\lambda_4, K_4, f_4), \text{i .e.}$$

$$\Big\{F\Big(\frac{\lambda_1}{\lambda_2}, K, \frac{f_1}{f_2}\Big) : f \le \frac{f_1 - f_2}{\hat{K}_2}\Big\} \cap \Big\{F\Big(\frac{\lambda_3}{\lambda_4}, K, \frac{f_3}{f_4}\Big) : f \le f_3 - \frac{\hat{K}_3}{\hat{K}_4} f_4\Big\} \neq \emptyset.$$

Thus there exist an operator $F(\lambda, K, f) \in \mathbb{G}_1$ such that

$$\lambda = \frac{\lambda_1}{\lambda_2} = \frac{\lambda_3}{\lambda_4} \quad \text{and} \quad K = \frac{K_1}{K_2} = \frac{K_3}{K_4}\,.$$

We have

$$\lambda_1\lambda_4 = \lambda_2\lambda_3, \quad K_1K_4 = K_2K_3$$

and f is a function satisfying

$$f \le \frac{f_1 - f_2}{\hat{K}_2}, \qquad f \le f_3 - \frac{\hat{K}_3}{\hat{K}_4} f_4\,.$$

Let us define

$$\lambda_\mu = \lambda_1\lambda_4 = \lambda_2\lambda_3, \quad K_\mu = K_1K_4 = K_2K_3$$

and

$$f_\mu \le \min\{\hat{K}_1 f_4 + f_1, \hat{K}_2 f_3 + f_2\}.$$

Then $F(\lambda_\mu, K_\mu, f_\mu) \in \mathbb{G}_1$ and with respect to Lemma 2 we have

$$F(\lambda_\mu, K_\mu, f_\mu) \in \{F(\lambda_1\lambda_4, K_1K_4, f) : f \ge f_\mu\} = F(\lambda_1, K_1, f_1) * F(\lambda_4, K_4, f_4),$$

$$F(\lambda_\mu, K_\mu, f_\mu) \in \{F(\lambda_2\lambda_3, K_2K_3, f) : f \ge f_\mu\} = F(\lambda_2, K_2, f_2) * F(\lambda_3, K_3, f_3),$$

consequently

$$F(\lambda_1, K_1, f_1) * F(\lambda_4, K_4, f_4) \approx F(\lambda_2, K_2, f_2) * F(\lambda_3, K_3, f_3),$$

hence the hypergroup $(\mathbb{G}_1, *)$ is a noncommutative join space. □

Thus $(\mathbb{G}_1, \odot)$ is a noncommutative group of Fredholm operators of the second kind which has a subgroups of the form

$$\mathbb{G}_0 = \{F(1, K, 0) : K \in C_+(J \times J)\},$$

$$\mathbb{G}_N = \{F(1, K, f) : K \in C_+(J \times J)\}.$$

The subgroup $(\mathbb{G}_0, \odot)$ of the group $(\mathbb{G}_1, \odot)$ is not normal, because

$$F(\lambda, K, f) \odot F(1, K_1, 0) \odot F^{-1}(\lambda, K, f) = \\ = F(\lambda, KK_1, f) \odot F\Big(\frac{1}{\lambda}, \frac{1}{K}, -\frac{f}{\hat{K}}\Big) = F\Big(1, K_1, -\hat{K}_1 f + f\Big),$$

and $\hat{K}_1$ need not be equal to the constant function with value 1.

The subgroup $(\mathbb{G}_N, \odot)$ of the group $(\mathbb{G}_1, \odot)$ is normal, because

$$F(\lambda, K, f) \odot F(1, K_1, f_1) \odot F^{-1}(\lambda, K, f) = \\ = F(\lambda, KK_1, \hat{K} f_1 + f) \odot F\Big(\frac{1}{\lambda}, \frac{1}{K}, -\frac{f}{\hat{K}}\Big) = F\Big(1, K_1, -\hat{K}_1 f + \hat{K} f_1 + f\Big).$$

Using these subgroups it is possible to construct the decompositions of $\mathbb{G}$ and $\mathbb{G}_1$ and to introduce hyperoperations on these decompositions. This leads to further interesting examples of join spaces—see [9].

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