Numerical range and numerical radius (An introduction)

K. F. Žitný, J. Kozánek*

In early studies of Hilbert spaces (by Hilbert, Hellinger, Toeplitz, and others) the object of chief interest were quadratic forms. Nowadays, quadratic questions about a linear continuous operator are questions about its numerical range - [1], [3], [5]. Students may find in this paper a motivation to go on to a deeper understanding of the properties of linear operators and matrices as well as some numerical methods how to determine their numerical ranges. It is rather surprising that the proof of the so called Elliptical Range Theorem for the matrices of order 2 is tedious - [4].

We deal with the Banach algebra $\mathbb{B}(H)$ of linear continuous operator on a complex Hilbert space H. The numerical range of an operator $T \in \mathbb{B}(H)$ is the subset of the complex field $\mathbb{C}$, given by

$$\mathcal{V}(T) = \{(Tx|x) \mid x \in H, \ ||x|| = 1\}.$$

The following properties of $\mathcal{V}(T)$ are immediate

$\mathcal{V}(\alpha I + \beta T) = \alpha + \beta \mathcal{V}(T)$ for $\alpha, \ \beta \in \mathbb{C}$,
$\mathcal{V}(T^*) = \{\overline{\lambda} \mid \lambda \in \mathcal{V}(T)\}$,
$\mathcal{V}(U^*TU) = \mathcal{V}(T)$ for any unitary $U \in \mathbb{B}(H)$.

The next fundamental result is known as the Toeplitz-Hausdorff theorem. (We shall not interrupt our presentation by defining every item of notation we use in this text.)

Theorem 1 The numerical range $\mathcal{V}(T)$ of $T \in \mathbb{B}(H)$ is convex.

Proof. Given $\lambda_1, \lambda_2 \in \mathcal{V}(T), \ \lambda_1 \neq \lambda_2$, we will prove that

$$(1-t)\lambda_1 + t\lambda_2 \in \mathcal{V}(T) \quad \text{whenever } t \in [0,1].$$

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*Institute of Thermomechanics AS CR, Dolejškova 5, 182 00 Prague 8, e-mail: kozanek@it.cas.cz

If $S = \alpha I + \beta T$, where $\alpha, \beta \in \mathbb{C}$ are such that $0 = \alpha + \beta\lambda_1$ and $1 = \alpha + \beta\lambda_2$, it is sufficient to show that $t \in \mathcal{V}(S)$ for all $t \in [0, 1]$.
Let us fix unit vectors $x, y \in H$ such that

$$0 = (Sx|x),\ 1 = (Sy|y)$$

and define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(t) = (Sx|y)\exp(-it) + (Sy|x)\exp(it),\ t \in \mathbb{R}.$$

Since $\cos\pi = -1$, it is obvious that $g(t + \pi) = -g(t)$ for every $t \in \mathbb{R}$. Moreover, there exists $t_0 \in [0, \pi]$ such that $\operatorname{Im} g(t_0) = 0$. Since $\operatorname{Im} g(0) = -\operatorname{Im} g(\pi)$, and g is a continuous function, there is $t_0 \in [0, \pi]$ such that $\operatorname{Im} g(t_0) = 0$.
Now observe that the vectors x and $\widehat{y} = \exp(it_0)y$ are linearly independent. Otherwise, $x = \alpha\ \widehat{y}$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and $0 = (Sx|x) = |\alpha|^2(S\widehat{y}|\widehat{y}) = (Sy|y) = 1$. To finish the proof, define continuous functions z and f by

$$z(s) = \frac{(1-s)x + s\widehat{y}}{||(1-s)x + s\widehat{y}||},\quad s \in [0, 1],$$

and

$$f(s) = (Sz(s)|z(s)),\quad s \in [0, 1].$$

A straightforward calculation shows that f is a real-valued function with $f(0) = 0$ and $f(1) = 1$. Thus $t \in [0, 1] \subset f([0, 1]) \subset [0, 1] \subset \mathcal{V}(S)$. ■

This theorem has many proofs, a recent one is due to C.K. Li „C-Numerical Ranges and C-Numerical Radii", Linear and Multilinear Algebra (1994), 37, 51-82. A short proof covering unbounded operators was given in K.E. Gustafson „The Toeplitz-Hausdorff Theorem of linear Operators", Proc. Amer. Math. Soc. (1970), 25, 203-204.

Lemma 1 An operator $T \in \mathbb{B}(H)$ is Hermitian iff $\mathcal{V}(T) \subset \mathbb{R}$.

Proof. If $T = T^*$, we have $(Tx|x) \in \mathbb{R}$ for every $x \in H$. Hence $\mathcal{V}(T) \subset \mathbb{R}$. Conversely, suppose $\mathcal{V}(T) \subset \mathbb{R}$. Then $((T - T^*)x|x) = (Tx|x) - (T^*x|x) = (Tx|x) - \overline{(Tx|x)} = 0$ for $x \in H$.

We give a brief account of some properties of the numerical radius.

The numerical radius of $T \in \mathbb{B}(H)$ is given by

$$v(T) = \sup\{|\lambda| \mid \lambda \in \mathcal{V}(T)\}.$$

Obviously, $v(T^*) = v(T)$ for every $T \in \mathbb{B}(H)$, and for any vector $x \in H$, we have

$$|(Tx|x)| \leq v(T) \cdot ||x||^2.$$

Lemma 2 For $T \in \mathbb{B}(H)$,

$$\frac{1}{2}||T|| \leq v(T) \text{ , and } v(T^2) \leq v(T)^2.$$

Proof. For $S \in \mathbb{B}(H)$ and unit vectors $x, y \in H$ we verify (with the help of the parallelogram identity) the following inequality:

$$\begin{aligned} |2(Sx|y) + 2(Sy|x)| &= |(S(x+y)|x+y) - (S(x-y)|x-y)| \\ &\leq v(S)(||x+y||^2 + ||x-y||^2) \\ &= 2v(S)(||x||^2 + ||y||^2) \\ &= 4v(s) \ . \end{aligned}$$

Taking $Sx \neq 0$ and $y = ||Sx||^{-1}Sx$, we conclude that

$$|||Sx||^2 + (S^2x|x)| \leq 2v(S)||Sx|| \ .$$

Let $s \in \mathbb{R}$ be such that $\exp(i2s)(T^2x|x) = |(T^2x|x)|$ and let $S = \exp(is)T$. Then

$$||Tx||^2 \leq ||Tx||^2 + |(T^2x|x)| \leq 2v(T)||Tx||,$$

and hence $||T|| \leq 2v(T)$.

Noting that

$$\begin{aligned} 0 &\leq 2v(T)||Tx|| - ||Tx||^2 - |(T^2x|x)| \\ &= -(v(T) - ||Tx||)^2 + v(T)^2 - |(T^2x|x)| \\ &\leq v(T)^2 - |(T^2x|x)| \ , \end{aligned}$$

we conclude that $|(T^2x|x)| \leq v(T)^2$, which implies $v(T^2) \leq v(T)^2$. ■

Observe that the numerical radius $v(T)$ is a norm on $\mathbb{B}(H)$ equivalent to the operator norm in view of the inequality $\frac{1}{2}||T|| \leq v(T) \leq ||T||$ valid for all $T \in \mathbb{B}(H)$.

Corollary 1

If $T \in \mathbb{B}(H)$ is normal, then $||T|| = v(T)$.

Proof. It is sufficient to show that $||T|| \leq v(T)$. By the previous lemma $v(T^2) \leq v(T)^2$ and one verifies easily by induction that $v(T^{2^k}) \leq v(T)^{2^k}$, $k \geq 1$. Since T is normal, we have $||T||^{2^k} = ||T^{2^k}|| \leq 2v(T^{2^k}) \leq 2v(T)^{2^k}$ and so $||T|| \leq \lim_{k\to\infty} 2^{\frac{1}{2^k}} v(T) = v(T)$. ∎

Theorem 2

For each $T \in \mathbb{B}(H)$ we have

$$v(T) = \max\{||\operatorname{Re}(\vartheta T)|| \mid \vartheta \in \mathbb{C},\ |\vartheta| = 1\}.$$

Proof. If $x \in H$ is a unit vector we have $|(Tx|x)| \geq |(\operatorname{Re} Tx|x)|$ and so $v(T) \geq v(\operatorname{Re} T) = ||\operatorname{Re} T||$ (because $\operatorname{Re} T$ is Hermitian). For $\vartheta \in \mathbb{C}$, $|\vartheta| = 1$ replacing T by ϑT we get

$$||\operatorname{Re}(\vartheta T)|| \leq v(T).$$

Suppose that

$$\varkappa = \sup\{||\operatorname{Re}(\vartheta T)|| \mid \vartheta \in \mathbb{C},\ |\vartheta| = 1\} < v(T),$$

then pick a unit vector $x \in H$ such that $\varkappa < |(Tx|x)|$, and for $\gamma = |(Tx|x)|$ put $\alpha = \gamma^{-1}(\operatorname{Re} Tx|x)$, $\beta = \gamma^{-1}(\operatorname{Im} Tx|x)$ and $\vartheta = \alpha - i\beta$. Then $|\vartheta|^2 = 1$, and $\operatorname{Re}(\vartheta T) = \operatorname{Re}((\alpha - i\beta)(\operatorname{Re} T + i \operatorname{Im} T)) = \alpha \operatorname{Re} T + \beta \operatorname{Im} T$. Consequently, $||\operatorname{Re}(\vartheta T)|| \geq (\operatorname{Re}(\vartheta T)x|x) = \alpha^2\gamma + \beta^2\gamma = \gamma = |(Tx|x)|$, a contradiction.
As the function $\vartheta \mapsto ||\operatorname{Re}(\vartheta T)|| = \frac{1}{2}||\vartheta T + \overline{\vartheta} T^*||$ is continuous, we conclude that

$$v(T) = \max\{||\operatorname{Re}(\vartheta T)|| \mid \vartheta \in \mathbb{C},\ |\vartheta| = 1\}. \ ∎$$

Remark. $\operatorname{Re} T = \frac{1}{2}(T + T^*)$, $(\operatorname{Re} T)^* = (\operatorname{Re} T)$; $\operatorname{Im} T = \frac{1}{2i}(T - T^*)$, $(\operatorname{Im} T)^* = -(\operatorname{Im} T)$.

Let us new look at two extreme cases of the inequalities for $v(T)$.

Lemma 3

Let $T \in \mathbb{B}(H)$.

a) If $v(T) = ||T||$, then there exists $\lambda \in \sigma_{ap}(T)$ such that $|\lambda| = ||T||$ and $r(T) = ||T||$,

b) If $\lambda \in \mathcal{V}(T)$, $|\lambda| = ||T||$, then $\lambda \in \sigma_p(T)$,

c) If $\mathcal{R}(T) \perp \mathcal{R}(T^*)$, then $v(T) = \frac{1}{2}||T||$,

where

$\sigma_p(T) = \{\lambda \in \mathbb{C} \,|\, \mathcal{N}(\lambda I - T) \neq \{0\}\}$, denotes the point spectrum of T,
$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} \,|$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors in H such that, $||(\lambda I - T)x_n|| \to 0\}$, the approximative spectrum,
$r(T) = \max\{|\lambda| \,|\, \lambda \in \sigma(T)\}$, the spectral radius.

Proof.

a) If $T \neq 0$, $v(T) = ||T|| = 1$, then there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of complex numbers such that $\lambda_n \in \mathcal{V}(T)$ and $1 - \frac{1}{n} < |\lambda_n| \leq 1$. Without loss of generality, we can suppose that $\lambda_n \to \lambda \in \overline{\mathcal{V}(T)}$. By construction, $\lambda_n = (Tx_n|x_n)$, with $x_n \in H$, $||x_n|| = 1$, and from the inequality $|(Tx_n|x_n)| \leq ||Tx_n|| \leq 1$, we get $\lim_{n\to\infty} ||Tx_n|| = 1$. Since $||(\lambda I - T)x_n||^2 = |\lambda|^2 - (Tx_n|\lambda x_n) - (\lambda x_n|Tx_n) + ||Tx_n||^2 = |\lambda|^2 - \overline{\lambda}(Tx_n|x_n) - \lambda(x_n|Tx_n) + ||Tx_n||^2 \to 2 - 2|\lambda|^2 = 0$.

b) By hypothesis, $\lambda = (Tx|x)$, $x \in H$, $||x|| = 1$. Then $||T|| = |\lambda| \leq ||Tx|| \leq ||T||$ and so $|(Tx|x)| = ||Tx||\,||x||$. Thus $Tx = \mu x$ for some $\mu \in \mathbb{C}$. However, $\lambda = (Tx|x) = \mu$, and hence $Tx = \lambda x$.

c) Let $x \in H$, $||x|| = 1$. If we write $x = y + z$ where $y \in \mathcal{N}(T)$, $z \in \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}$, then $(Tx|x) = (Tz|y+z) = (Tz|y)$ and $|(Tx|x)| \leq ||Tz||\,||y|| \leq ||T||\,||y||\,||z|| \leq ||T||\frac{||y||^2+||z||^2}{2} = \frac{1}{2}||T||$. Since x is arbitrary, we have $v(T) \leq \frac{1}{2}||T|| \leq v(T)$. ■

Theorem 3

Let $T \in \mathbb{B}(H)$ and $v(T) \leq 1$. Then $v(T^n) \leq 1$ for all $n \in \mathbb{N}$.

Using the well-known identity

$$(1 - \mu^n)^{-1} = \frac{1}{n}\sum_{k=0}^{n-1}(1 - \omega^k \mu)^{-1}, \; \mu \in \mathbb{C}, \; |\mu| < 1,$$

where $\omega = \exp\frac{2\pi i}{n}$, which obviously holds when μ is replaced by operators λT, $\lambda \in \mathbb{C}$, $|\lambda| < 1$, we write

$$(1-\lambda^n T^n)^{-1} = \frac{1}{n}\sum_{k=0}^{n-1}(1-\omega^k\,\lambda T)^{-1}.$$

Since $\mathrm{Re}((I-\omega^k\,\lambda T)y\,|\,y) = ||y||^2 - \mathrm{Re}(\omega^k\,\lambda\,(Ty\,|\,y)) \geq ||y||^2(1-|\lambda|) \geq 0,\ y \in H$, we have $\mathrm{Re}((I-\omega^k\,\lambda T)^{-1}x\,|\,x)$ for all $x \in H$, writing $y = (I-\omega^k\,\lambda T)^{-1}x$.

Hence

$\mathrm{Re}((I-\lambda^n\,T^n)^{-1}x\,|\,x)) \geq 0$ and also $\mathrm{Re}((I-\lambda^n\,T^n)x\,|\,x)) \geq 0$ for all $x \in H$, and $\lambda \in \mathbb{C}$, $|\lambda| < 1$.

Setting $\lambda = s\exp it$, $0 < s < 1$, $t \in \mathbb{R}$ and then letting $s \nearrow 1$, we deduce that $||x||^2 \geq \mathrm{Re}(\exp i\,n\,t\,T^n x\,|\,x)$ and hence $v(T^n) \leq 1$. ■

In the following H denotes a Hermitian space. That is,a finite-dimensional Hilbert space over $\mathbb{C}$. Let $\mathcal{L}(H)$ denotes the algebra of linear operators on H. If an operator $T \in \mathcal{L}(H)$ is represented by a matrix, we always assume that the corresponding basis of H is orthonormal. By $\mathcal{K}(\mathbb{C})$ we denote the metric space of nonempty compact subsets of $\mathbb{C}$ endowed with the Hausdorff metric Δ.

If $K_1, K_2 \in \mathcal{K}(\mathbb{C})$, then

$$\Delta(K_1,K_2) = \max[\sup\{dist(\lambda,K_1)\,|\,\lambda \in K_2\}, \sup\{dist(\lambda,K_2)\,|\,\lambda \in K_1\}]$$

where $dist(\lambda, K_j) = \inf\{|\lambda-\mu|\,|\,\mu \in K_j\}$, $j = 1, 2$.

The most important facts about $\mathcal{V}(T)$ are the following.

Theorem 4

a) If $T \in \mathcal{L}(H)$, then $\mathcal{V}(T)$ is a compact, convex subset of $\mathbb{C}$, and the numerical radius is attained.

b) $\sigma(T) \subset \mathcal{V}(T)$ for every $T \in \mathcal{L}(H)$.

c) $\mathcal{V}(T+S) \subset \mathcal{V}(T) + \mathcal{V}(S)$, $T,\ S \in \mathcal{L}(H)$.

d) $\Delta(\mathcal{V}(T),\ \mathcal{V}(S)) \leq ||T-S||$, $T,\ S \in \mathcal{L}(H)$.

Proof.

a) We already know that the numerical range is convex. The function $x \mapsto (Tx|x)$ from the compact set $\{x \in H\,|\,||x|| = 1\}$ into $\mathbb{C}$ is continuous.

Since the continuous image of a compact set is compact, the compactness of $\mathcal{V}(T)$ is obvious. Further, the real valued function $x \mapsto |(Tx|x)|$ from S attains its maximal value.

b) For any $\lambda \in \sigma(T)$, we have $\lambda x = Tx,\ x \in H,\ ||x|| = 1$, and $\lambda = (\lambda x|x) = (Tx|x)$.

c) $\mathcal{V}(T+S) \subset \{(Tx|x) \,|\, x \in H,\ ||x|| = 1\} + \{(Sx|x) \,|\, x \in H,\ ||x|| = 1\} = \mathcal{V}(T) + \mathcal{V}(S)$.

d) Given $\mu \in \mathcal{V}(T)$ choose a unit vector $x \in H$ such that $\mu = (Tx|x)$. Setting $\beta = (Sx|x)$ yields $|\mu - \beta| = |(T-S)x|x| \leq ||T-S||$. Accordingly $dist(\mu, \mathcal{V}(S)) = \inf\{|\mu-\gamma| \,|\, \gamma \in \mathcal{V}(S)\} \leq ||T-S||$, and $\sup\{dist(\mu, \mathcal{V}(S)) \,|\, \mu \in \mathcal{V}(T)\} \leq ||T-S||$. By symmetry, we conclude that $\Delta(\mathcal{V}(T), \mathcal{V}(S)) \leq ||T-S||$. ■

Remarks

1) If $T,\ S \in \mathcal{L}(H)$, then $\sigma(T+S) \subset \mathcal{V}(T+S) \subset \mathcal{V}(T) + \mathcal{V}(S)$.

2) We say that $T \to \mathcal{V}(T)$ is continuous at $T_0 \in \mathcal{L}(H)$ if $\lim_{n\to\infty} \Delta(\mathcal{V}(T_n), \mathcal{V}(T_0)) = 0$ for every sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(H)$ converging to T_0.

Lemma 4

Let $T \in \mathcal{L}(H)$. If $\lambda \in \mathbb{C}$ and $d = dist(\lambda, \mathcal{V}(T)) > 0$, then $\lambda \in \rho(T)$ and

$$||(\lambda I - T)^{-1}|| \leq d^{-1}.$$

Proof. For $x \in H,\ ||x|| = 1$, we have $||(\lambda I - T)x|| \geq |((\lambda I - T)x|x)| = |\lambda - (Tx|x)| \geq d$. Suppose $\mathcal{N}(\overline{\lambda} I - T^*) \neq \{0\}$, and choose $y \in H,\ ||y|| = 1$ such that $T^*y = \overline{\lambda} y$. Then $\lambda = (y|\overline{\lambda} y) = (y|T^*y) = (Ty|y) \in \mathcal{V}(T)$, a contradiction. Hence $\lambda I - T \in \mathcal{GL}(H)$. Given $y \in H$ let $x = (\lambda I - T)^{-1}y$. Since $||(\lambda I - T)x|| \geq d||x||$ we have $||y|| \geq d\,||(\lambda I - T)^{-1}y||$ or $d^{-1} \geq ||(\lambda I - T)^{-1}||$. ■

Corollary 2

If $T \in \mathcal{L}(H)$ assume that there exists $\omega \geq 0$ such that $\mathrm{Re}(Tx|x) \leq -\omega$ for every $x \in H,\ ||x|| = 1$. Then

$$||(\lambda I - T)^{-1}|| \leq \frac{1}{\omega + \mathrm{Re}\,\lambda}$$

for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$.

Proof. It is sufficient to observe that $|\lambda - (Tx|x)| \geq |\mathrm{Re}\,\lambda - \mathrm{Re}(Tx|x)| \geq \mathrm{Re}\,\lambda + \omega$ for $x \in H,\ ||x|| = 1$ and $\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda > 0$. ■

Examples

In the following we use the matrix representation of $T \in \mathcal{L}(\ell_2^2)$ with respect to the standard basis of ℓ_2^2.

1) Let $T \in \mathcal{L}(\ell_2^2)$ be represented by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

If $x = (\alpha_1, \alpha_2) \in \ell_2^2$, $||x|| = |\alpha_1|^2 + |\alpha_2|^2 = 1$, then $Tx = (\alpha_2, 0)$ and $|(Tx|x)| = |\alpha_1\alpha_2| \le \frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2) = \frac{1}{2}$ and thus $\mathcal{V}(T) \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| \le \frac{1}{2}\}$. Given $\lambda = r \exp i\varphi$, $0 \le r \le \frac{1}{2}$, $\varphi \in \mathbb{R}$,we consider the unit vector $x = (\cos\alpha, (\exp i\varphi)\sin\alpha)$ with $\alpha = \frac{1}{2}\arcsin 2r$. Since $(Tx|x) = \exp i\varphi \sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha \exp i\varphi = r\exp i\varphi$, we see that $\mathcal{V}(T) = \{\lambda \in \mathbb{C} \,|\, |\lambda| \le \frac{1}{2}\}$ and $v(T) = \frac{1}{2}$.

If $T \in \mathcal{L}(\ell_2^2)$ is given by $\begin{bmatrix} 0 & \exp i\psi \\ 0 & 0 \end{bmatrix}$, $\psi \in \mathbb{R}$, we note that $\mathcal{V}(S) = \mathcal{V}(T)$.

2) Let $T \in \mathcal{L}(\ell_2^2)$ be represented by

$$\begin{bmatrix} -1 & 2d \\ 0 & 1 \end{bmatrix}, d > 0 .$$

Then $\mathcal{V}(T)$ is an ellipse (with its interior) with the center at the origin, the foci of which are the eigenvalues of T.

Any unit vector $x \in \ell_2^2$ may be written in the form

$$x = \exp i\psi(\cos\vartheta,\ \exp i\varphi \sin\vartheta), \text{ where } \vartheta \in [0, 2\pi[,\ \varphi, \psi \in \mathbb{R} .$$

Since $(Tx|x) = (Ty|y)$ if $y = (\exp i\tau)x$, $\tau \in \mathbb{R}$, we can assume that $x = (\cos\vartheta,\ \exp i\varphi\sin\vartheta)$. Then $(Tx|x) = -\cos 2\vartheta + d\exp i\varphi \sin 2\vartheta$ and so $|(Tx|x) + \cos 2\vartheta|^2 = d^2\sin^2 2\vartheta$. Write $(Tx|x) = u + iv$, where u, $v \in \mathbb{R}$, and observe that

$$(u + \cos 2\vartheta)^2 + v^2 = d^2(1 - \cos^2 2\vartheta) .$$

Since $0 \le (\cos 2\vartheta + \frac{u}{1+d^2})^2 = \frac{u^2}{(1+d^2)^2} + \frac{d^2 - (u^2+v^2)}{1+d^2}$, it follows easily that

$$\frac{u^2}{1+d^2} + \frac{v^2}{d^2} \le 1 .$$

In other words, if $\lambda \in \mathcal{V}(T)$, the point $(\mathrm{Re}\,\lambda,\ \mathrm{Im}\,\lambda)$ lies in the elliptical disc with center $(0,\ 0)$, minor axis $2d$, and major axis $2\sqrt{1+d^2}$.

If $(u, v) \in \mathbb{R}^2$ and $\frac{u^2}{1+d^2} + \frac{v^2}{d^2} = 1$, we shall show that $u + iv \in \mathcal{V}(T)$. To see it, choose $t \in [0, 2\pi[$ such that

$$u = \sqrt{1+d^2}\cos t, \quad v = d\sin t \ ,$$

and set $x(t) = \frac{1}{\sqrt{\varkappa}}(d\cos t - i\sqrt{1+d^2}\sin t,\ \cos t + \sqrt{1+d^2})$, where $\varkappa = 2\sqrt{1+d^2}(\sqrt{1+d^2} + \cos t)$. Then $x(t) \in \ell_2^2$ is a unit vector, and we have

$$(Tx(t)|x(t)) = \sqrt{1+d^2}\cos t + id\sin t$$

as desired.

Since $\mathcal{V}(T)$ is convex, its points fill up the ellipse, the foci of which have the coordinates $(-1, 0)$ and $(1, 0)$.

3) Let $T \in \mathcal{L}(\ell_2^2)$ be represented by

$$\begin{bmatrix} -1 & d\exp i\varphi \\ 0 & 1 \end{bmatrix}, \quad d \in \mathbb{R}^*, \quad \varphi \in \mathbb{R} \ .$$

Then $\mathcal{V}(T) = \mathcal{V}(S)$, where $S \in \mathcal{L}(\ell_2^2)$ is given by the matrix

$$\begin{bmatrix} -1 & |d| \\ 0 & 1 \end{bmatrix} .$$

Choose $\psi \in \mathbb{R}$ such that $d\exp i\varphi = |d|\exp i\psi$. If $x = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ is a unit vector, and $y = (\lambda_1, \lambda_2 \exp i\psi) \in \mathbb{C}^2$. A straightforward calculation shows that

$$(Tx|x) = (Sy|y) \ .$$

4) Let $0 < b \leq a$. If $T \in \mathcal{L}(\ell_2^2)$ is given (with respect to the standard basis) by

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$$

then $\mathcal{V}(T)$ is an ellipse (possibly degenerate) with the interior, and it is centered at the origin. Its minor axis is along the imaginary axis and has length $a - b$; its major axis is along the imaginary axis and has length $a + b$. The foci are at $\pm\sqrt{ab}$.

Since $(Tx|x) = (T(\exp i\varphi)x|(\exp i\varphi)x)$ for any $\varphi \in \mathbb{R}$, $x \in \ell_2^2$, to determine $\mathcal{V}(T)$ it suffices to consider $(Tx|x)$ for unit vectors x whose first component is real and nonnegative.

Let us consider the vectors $x(t) = (t, (1-t^2)^{\frac{1}{2}} \exp i\vartheta)$, $0 \le t \le 1$, $0 \le \vartheta \le 2\pi$. A calculation shows that $(Tx(t)|x(t)) = t(1-t^2)^{\frac{1}{2}}[(a+b)\cos\vartheta + i\sin\vartheta]$. As ϑ varies 0 to 2π, and $t = \frac{1}{\sqrt{2}}$, the point $\frac{1}{2}[(a+b)\cos\vartheta + i\sin\vartheta]$ traces out a possible degenerate ellipse centered at origin. Its foci are located on the major axis at distance $((\frac{a+b}{2})^2 - (\frac{a-b}{2})^2)^{\frac{1}{2}} = \sqrt{ab}$ from the center.

5) Let $T \in \mathcal{L}(\ell_2^2)$ be represented by an upper triangular matrix

$$\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{bmatrix}, \ \lambda_1,\ \lambda_2,\ \alpha \in \mathbb{C}.$$

a) If $\lambda = \lambda_1 = \lambda_2$, then T is represented by

$$|\alpha| \begin{bmatrix} 0 & \exp i\varphi \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \varphi = Arg\alpha\ ,$$

and $\mathcal{V}(T) = |\alpha|\{\gamma \in \mathbb{C}\,|\,|\gamma| \le \frac{1}{2}\} + \{\lambda\}$. In particular, $\mathcal{V}(T)$ is a circular disc with center λ and radius $\frac{1}{2}|\alpha|$, if $\alpha \ne 0$; otherwise $\mathcal{V}(T) = \{\lambda\}$.

b) If $\lambda_1 \ne \lambda_2$ and $\alpha = 0$, we consider the matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

If $x = (\alpha_1, \alpha_2) \in \ell_2^2$ is a unit vector, then $(Tx|x) = \lambda_1|\alpha_1|^2 + \lambda_2|\alpha_2|^2 = t\lambda_1 + (1-t)\lambda_2$, $t \in [0,1]$. So $\mathcal{V}(T)$ is the line segment joining λ_1 and λ_2.

c) If $\lambda_1 \ne \lambda_2$ and $\alpha \ne 0$, then T is given by

$$\frac{\lambda_2 - \lambda_1}{2} \begin{bmatrix} -1 & \frac{2\alpha}{\lambda_2 - \lambda_1} \\ 0 & 1 \end{bmatrix} + \frac{\lambda_1 + \lambda_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $S \in \mathcal{L}(\ell_2^2)$ is represented by $\frac{|\lambda_2 - \lambda_1|}{2} \begin{bmatrix} -1 & \frac{2\alpha}{\lambda_2 - \lambda_1} \\ 0 & 1 \end{bmatrix}$, then (cf. Examples 2 and 3) $\mathcal{V}(S)$ is an ellipse (with its interior) centered at the origin. The length of its minor (respectively major) axis is $|\alpha|$ (respectively $\sqrt{|\lambda_1 - \lambda_2|^2 + |\alpha|^2}$). Observe that $|\alpha| = \sqrt{Tr(T^*T) - |\lambda_1|^2 - |\lambda_2|^2}$.

Therefore

$$\mathcal{V}(T) = \left(\frac{\lambda_1 + \lambda_2}{2}\right) + \exp i\psi \cdot \mathcal{V}(S), \text{ where } \psi = Arg(\lambda_2 - \lambda_1),$$

is an ellipse centered at $\frac{\lambda_1+\lambda_2}{2}$ with foci at λ_1, λ_2; the major axis has an inclination of ψ with the real axis.
If $x_1, x_2 \in \ell_2^2$ are unit eigenvectors of T, $Tx_1 = \lambda_1 x_1$, $Tx_2 = \lambda_2 x_2$, the eccentricity of the ellipse is $\sin\vartheta$, where ϑ is reduced angle between the eigenvectors, that is, $\vartheta = \arccos|(x_1|x_2)|$.
Let $x_1 = (1,0)$, $x_2 = \frac{1}{\sqrt{|\alpha|^2+|\lambda_1-\lambda_2|^2}}(\alpha, \lambda_2 - \lambda_1)$. Clearly, x_1, x_2 are unit eigenvectors of T corresponding λ_1 and λ_2, respectively, and $|(x_1|x_2)| = \frac{1}{\sqrt{|\alpha|^2+|\lambda_1-\lambda_2|^2}} = \cos\vartheta$, $0 < \vartheta < \frac{\pi}{2}$. Hence $\sin\vartheta = \sqrt{1-\cos^2\vartheta} = \frac{|\lambda_1-\lambda_2|}{\sqrt{|\alpha|^2+|\lambda_1-\lambda_2|^2}}$.

Coda

If $T \in \mathcal{L}(H)$, $\dim H = 2$, then the shape of $\mathcal{V}(T)$ can be easily extracted from an examination of the previous example.

Without loss of generality we can suppose that T is represented (with respect to an orthonormal basis of H) by an upper triangular matrix

$$\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1, \ \lambda_2, \ \alpha \in \mathbb{C}.$$

Thus $\mathcal{V}(T)$ is an ellipse (with interior) whose foci are the eigenvalues of T. Furthermore, $\mathcal{V}(T)$ is a segment (possibly degenerate) iff T is normal. If T is normal, then $\alpha = 0$ and $(Tx|x) = t\lambda_1 + (1-t)\lambda_2$ for any unit vector $x \in H$. Reciprocally, if $\mathcal{V}(T)$ is a segment or a point, then $\alpha = 0$. Therefore, if T satisfies $bd(\mathcal{V}(T)) \cap \sigma(T) \neq \emptyset$, then T is normal. In fact, since $\lambda_1 \in bd(\mathcal{V}(T))$, respectively $\lambda_2 \in bd(\mathcal{V}(T))$, the ellipse must be degenerate.

Computation of the $\mathcal{V}(T)$

Here, we suppose that T is a linear operator represented by the matrix $T \in \mathbb{C}^{n,n}$. The location of the $\mathcal{V}(T)$ in the complex plane is possible -[3] via Gersgorin-type inclusion, where $\mathcal{V}(T)$ lies in the convex hull of some Gersgorin's circular discs. The other numerical technique is to generate an enough large number of random unit vectors and to draw the points $(Tx|x)$ in the complex plane. More effective is the following approximation method - [2, 3]. Because the $\mathcal{V}(T)$ is convex and compact, it suffices to determine the boundary $bd(\mathcal{V}(T))$ of $\mathcal{V}(T)$. The general strategy is to calculate many well-spaced points on $bd(\mathcal{V}(T))$ and (or) many support lines of $\mathcal{V}(T)$ in these points. The convex hull of these boundary points is then a convex polygonal approximation to $\mathcal{V}(T)$, while the intersection of the half-spaces determined by the support lines of $\mathcal{V}(T)$ will be a convex polygonal approximation to $\mathcal{V}(T)$ that contains $\mathcal{V}(T)$. The area of the region between these two convex polygonal approximations may be thought of as a measure of how well is approximated $\mathcal{V}(T)$. The usefulness of the operator $\operatorname{Re} T = \frac{1}{2}(T + T^*)$ for the investigation of the $\mathcal{V}(T)$ will be shown in the consecutive lemmas.

Lemma 5 $\mathcal{V}(\operatorname{Re} T) = \operatorname{Re} \mathcal{V}(T)$.

Proof. A calculation gives: $(\operatorname{Re}(T)x|x) = \frac{1}{2}((Tx|x)+(T^*x|x)) = \frac{1}{2}((Tx|x)+(Tx|x)^*) = \operatorname{Re}(Tx|x)$ and each point in $\mathcal{V}(\operatorname{Re} T)$ is of the form $\operatorname{Re} z$ for some $z \in \mathcal{V}(T)$ and vice versa. ■

Lemma 6

If $T \in \mathbb{C}^{n,n}$, $x \in \mathbb{C}^n$, $(x|x) = 1$, then the following three conditions are equivalent:

a) $\operatorname{Re}(Tx|x) = \max_{\alpha \in \mathcal{V}(T)} \operatorname{Re} \alpha$

b) $(\operatorname{Re}(T)x|x) = \max_{r \in \mathcal{V}(\operatorname{Re} T)} r$

c) $(\operatorname{Re} T)x = \lambda_{\max}(\operatorname{Re} T)x$

where $\lambda_{\max}(\operatorname{Re} T)$ denotes the largest eigenvalue of the Hermitian matrix $\operatorname{Re} T$.

Proof. The equivalence of a) and b) follows from the identity $\operatorname{Re}(Tx|x) = \frac{1}{2}((Tx|x) + (T^*x|x)) = (\operatorname{Re}(T)x|x)$ and from the previous lemma. Labeling n orthonormal eigenvalues and corresponding eigenvectors of the Hermitian matrix $\operatorname{Re} T$ as x_j, λj, $j = 1, 2, ..., n$, then $x \in \mathbb{C}^n$, $(x|x) = 1$ may be written

$$x = \sum_{j=1}^{n} c_j x_j, \text{ where } \sum_{j=1}^{n} |c_j|^2 = 1.$$

Then $(\operatorname{Re} Tx|x) = \sum_{j=1}^{n} |c_j|^2 \lambda_j$ and the equivalence of b) and c) follows from extremal properties of eigenvalues. ■
The above lemma says that

$$\max_{\alpha \in \mathcal{V}(T)} \operatorname{Re} \alpha = \max_{r \in \mathcal{V}(\operatorname{Re} T)} r = \lambda_{\max}(\operatorname{Re} T)$$

i.e. that the furthest point "to the right - in the positive direction of Re axis" in $\mathcal{V}(T)$ is $\lambda_{\max}(\operatorname{Re} T) = \lambda_n$ and that $x = x_n$, where x_n is corresponding unit eigenvector. Computing eigenvalue $\lambda_{\max}(\operatorname{Re} T)$ and corresponding eigenvector x, $(x|x) = 1$ we obtain the boundary point $(Tx|x)$ of $\mathcal{V}(T)$ and the support line $L = \{\lambda_{\max}(\operatorname{Re} T) + t\,i \,|\, t \in \mathbb{R}\}$ of the convex set $\mathcal{V}(T)$ in this point. Support line L contains at least one point from $bd(\mathcal{V}(T))$, and $\mathcal{V}(T)$ is contained in one of the closed half-planes defined by L, i.e. $\mathcal{V}(T)$ lies in the half-plane

$$H = \{z \,|\, \operatorname{Re} z \leq \lambda_{\max}\}.$$

To obtain other boundary points and other support lines observe that:

$$\exp(-\,i\,\theta)\mathcal{V}(\exp(i\,\theta)T) = \mathcal{V}(T)$$

for all $\theta \in \mathbb{R}$. We will rotate $\mathcal{V}(T)$ and for each θ we calculate the eigenvalue

$$\lambda_\theta = \lambda_{\max}(\operatorname{Re}(\exp(i\,\theta)T))$$

and associated eigenvector $x_\theta \in \mathbb{C}^n$, $(x_\theta|x_\theta) = 1$. Then we define the line

$$L_\theta = \{\exp(-\,i\,\theta)(\lambda_\theta + t\,i) \,|\, t \in \mathbb{R}\}$$

and the half-plane

$$H_\theta = \exp(-\,i\,\theta)\{z \,|\, \operatorname{Re} z \leq \lambda_\theta\}\ . \ \blacksquare$$

In the following we shall use the above notation.

Theorem 5

For a matrix $T \in \mathbb{C}^{n,n}$ and each $\theta \in [0, 2\pi)$, the complex number $p_\theta = (Tx_\theta|x_\theta)$ is a boundary point of $\mathcal{V}(T)$. The line L_θ is a support line of $\mathcal{V}(T)$, with $p_\theta \in L_\theta \cap \mathcal{V}(T)$ and $\mathcal{V}(T) \subset H_\theta$.

Proof. From geometrical insight follows that each extreme point of $\mathcal{V}(T)$ occurs as a p_θ and that for any $\alpha \notin \mathcal{V}(T)$ there is an L_θ separating $\mathcal{V}(T)$ and α, i.e. $\alpha \notin H_\theta$. $\blacksquare$

Thus, we can represent $\mathcal{V}(T)$ with the help of the next theorem.

Theorem 6

For the matrix $T \in \mathbb{C}^{n,n}$

$$\mathcal{V}(T) = Conv(\{p_\theta \,|\, 0 \leq \theta < 2\pi\}) = \bigcap_{0 \leq \theta < 2\pi} H_\theta\ .$$

In practice, we work with the discrete analog of the above theorem. We define Θ as a set of angular mesh points

$$\Theta = \{\theta_1, \theta_2, ..., \theta_k\}, \quad 0 \leq \theta_1 < \theta_2 < ... < \theta_k < 2\pi$$

and we calculate for each $\theta_2 \in \Theta$ the point p_{θ_i} and we define the half space H_{θ_i} and support line L_{θ_i}. Then the inner and outer approximating sets for $\mathcal{V}(T)$ are

$$\mathcal{V}_{In}(T, \Theta) = Conv(\{p_{\theta_1}, ..., p_{\theta_k}\}) \text{ and } \mathcal{V}_{Out}(T, \Theta) = H_{\theta_1} \cap ... \cap H_{\theta_k}$$

and for every angular mesh Θ is

$$\mathcal{V}_{In}(T, \Theta) \subset \mathcal{V}(T) \subset \mathcal{V}_{Out}(T, \Theta) \ .$$

Together with the each point p_{θ_i} we can define the point q_{θ_i} as the (finite) intersection point of the lines L_{θ_i} and $L_{\theta_{i+1}}$, where $i = 1, ..., k$ and $i = k+1$ is identified with $i = 1$. Then

$$\mathcal{V}_{Out}(T, \Theta) = H_{\theta_1} \cap ... \cap H_{\theta_k} = Conv(\{q_{\theta_1}, ..., q_{\theta_k}\}) \ .$$

The polygonal approximations of the $\mathcal{V}(T)$ are

$$Conv(\{p_{\theta_1}, ..., p_{\theta_k}\}) \subset \mathcal{V}(T) \subset Conv(\{q_{\theta_1}, ..., q_{\theta_k}\}) \ .$$

On the Fig. 1 is numerical range $\mathcal{V}(T_1)$ and its polygonal approximations $\mathcal{V}_{In}(T_1, \Theta) = Conv(\{p_{\theta_1}, ..., p_{\theta_k}\})$, and $\mathcal{V}_{Out}(T_1, \Theta) = Conv(\{q_{\theta_1}, ..., q_{\theta_k}\})$ in the complex plane. The eigenvalues of the matrix

$$T_1 = \begin{bmatrix} 1+i & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

are marked by ($\circledast$).

The difference of their areas (or some other measure of their set difference) may be taken as a measure of the quality of the above approximation.

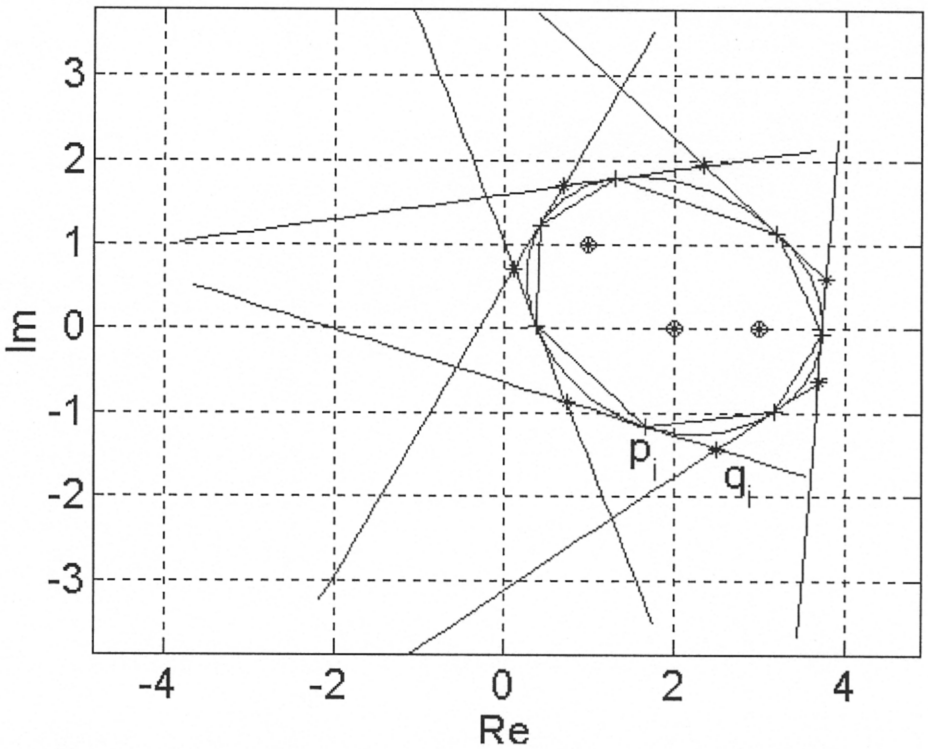

Fig. 1 $\mathcal{V}(T_1)$ and polygonal approximations $\mathcal{V}_{In}(T_1, \Theta)$ and $\mathcal{V}_{Out}(T_1, \Theta)$

Corollary 3

For the matrix $T \in \mathbb{C}^{n,n}$ and every angular mesh Θ, we can approximate the numerical radius $v(T)$ as

$$\max_{1 \le i \le k} |p_{\theta_i}| \le v(T) \le \max_{1 \le i \le k} |q_{\theta_i}| \ .$$

We note that numerical range $\mathcal{V}(T)$ and numerical radius $v(T)$ of a complex matrix T are arbitrarily closely approximated by calculating a series of the eigenvalue problems of Hermitian matrices only.

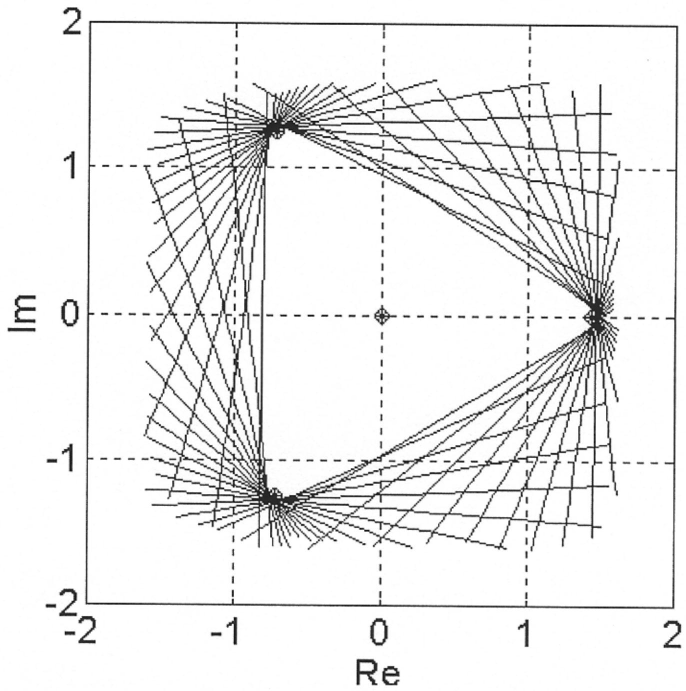

Fig. 2 $\mathcal{V}(T_2)$ approximated by support lines L_{θ_i} , $i = 1, ..., 50$

On the Fig. 2 is the approximation of $\mathcal{V}(T_2)$ by support lines L_{θ_i} , $i = 1, ..., 50$ in the complex plane. The eigenvalues of the matrix

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

are marked by ($\circledast$). The detail of the boundary of $\mathcal{V}(T_2)$ in the neighbourhood of the real eigenvalue $3^{\frac{1}{3}}$ of T_2 is on the Fig.3.

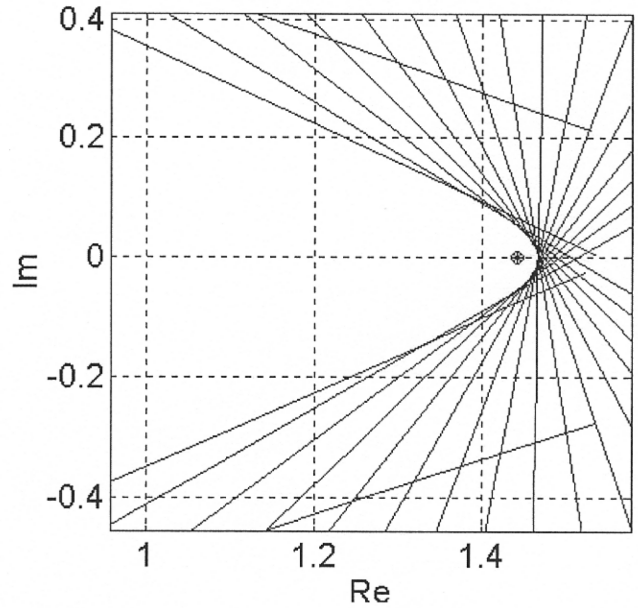

Fig. 3 Detail of the boundary of $\mathcal{V}(T_2)$ from the Fig. 2

On the Fig. 4 is the approximation of polygonal numerical range $\mathcal{V}(T_3)$ by support lines L_{θ_i} , $i = 1, ..., 100$ in complex plane. The eigenvalues of the normal matrix

$$T_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

are marked by ($\circledast$).

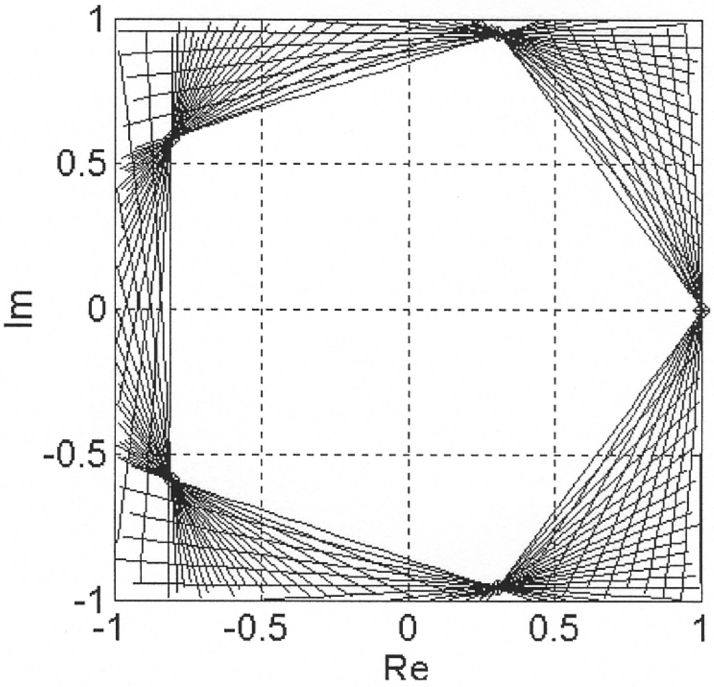

Fig. 4 Approximation of polygonal $\mathcal{V}(T_3)$ by support lines L_{θ_i} , $i = 1, ..., 100$

Notation

H the Hilbert space over $\mathbb{C}$
$\mathbb{B}(H)$ the Banach algebra of continuous linear operators on H
I the identity operator
$\mathbb{N}, \mathbb{R}, \mathbb{C}$ the field of natural, real, complex numbers
$(x|y)$ the scalar product of vectors x, $y \in H$
$||x||$ the Euclidean norm, $(x|x) = ||x||^2$
$\mathcal{V}(T)$ the numerical range of an operator $T \in \mathbb{B}(H)$
T^* the adjoint operator to T
$v(T)$ the numerical radius of $T \in \mathbb{B}(H)$
$||T||$ the norm of the operator T
$\ell_2^2 = \ell_2^2(\mathbb{C})$ the two-dimensional Hilbert space over $\mathbb{C}$ whose elements are vectors $(\alpha_1, \alpha_2) \in \mathbb{C}^2$
$\mathrm{Re}\, T = \frac{1}{2}(T + T^*)$, $\mathrm{Im}\, T = \frac{1}{2i}(T - T^*)$
$\sigma(T)$ the spectrum of T

$\sigma_p(T)$ the point spectrum of T
$\sigma_{ap}(T)$ the approximative point spectrum of T
$\mathcal{R}(T)$ the range of T
$\mathcal{N}(T)$ the null space of T
$\mathcal{L}(H)$ the algebra of linear operators on H, $\dim H < +\infty$
$\mathcal{GL}(H)$ the linear group of invertible linear operators on H, $\dim H < +\infty$
$\dim H$ the dimension of the space H
$bd(\mathcal{V}(T))$ the boundary of $\mathcal{V}(T)$
$T \in \mathbb{C}^{n,n}$ the complex matrix of order n
$\lambda_{\max}(\mathrm{Re}\, T)$ the maximum eigenvalue of Hermitian operator
$\mathrm{Re}\, T$, $T \in \mathcal{L}(H)$

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