Mathematical Methods in Geometric Modelling

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1. Introduction - development of geometric modellling

Computer aided geometric modelling has developed within the last 36 years from the simple tools enabling sketching and drawing (Sketchpad - Sutherland, 1963) up to interactive tools for modelling and visualisation based on the parametric and variational models, or today also on fuzzy and fractal principles of modelling chaotic behaviour. In the history of development of program systems, the appearance of free-form modelling (Coons and Ferguson at about 1964 - 1965) played a crucial role, bringing new methods of approximation and interpolation to the geometry. The first automatic system DAC - 1, developed in the USA in General Motors in 1965, provided complete technical documentation of the production, partial algorithmization of the data structure and possibility of modelling curves and surfaces. Fast development of the car and aircraft industry and construction of NC-machines influenced also the development of the modelling methods. This appeared on the development of new commercial graphical packages, graphical support for the programming languages and on the development of the self-standing graphical programming languages. Recently there appeared special computer algebras, packages for the mathematically - geometrically oriented user environment.

2. Geometric modelling

Term "geometric modelling" appeared at the beginning of 70's together with the dynamic development of the Computer Graphics and new techniques of modelling and production based on the NC-machines. It is related to the set of methods used for definitions of the form and other geometric characteristics of objects. Geometric modelling comprises a wide range of disciplines connected to the representation of geometric objects, from so called computational geometry to robust object-oriented systems of modelling and deformations of figures and processes based on the variational principles. It can be summarised that geometric modelling presents an elegant synthesis of geometry and computer, which can utilise advantage and possibilities of both disciplines. Methods of geometric modelling are used for construction of precise mathematical description of a real object or a simulation of some process. This construction is usually a set of specific operations performed by computer, which are necessary for creating a model of a real object, its storing in a database, analysis, modifications a visualisations. Methods of geometric modelling are formed as synthesis of principles and algorithms from various disciplines, parts of mathematics as e.g. analytic (coordinate) and descriptive geometry, topology, set theory, differential geometry, numerical analysis, linear and vector algebra, functional analysis, and others.

Generally we can speak about three basic aspects of geometric modelling.

1. representation, or mathematical model of the object that is really existing or non-existing and designed, or a simulation of a process or event in the form of mathematical formulae (formulas, equations, algorithms, or data structures as graphs, diagrams, relations, mappings and views, and others) based on the principles of the desired physical shape and determined properties of the modelled object, as suitable approximation through one of the possible representations.

2. modelling, finding an optimal shape of the modelled object satisfying the given functional and aesthetic requirements by variation of parameters in the mathematical model with the possible interactive control by visualisation of model.
3. rendering, graphical processing of the image to provide a realistic visualisation of the object model with the required layout, visibility, enlightenment, shading and luminosity.

Architecture of the geometric modelling system is composed from several components:

- software for geometric modelling (algorithms, relations, representations, incidence structure, equations of figures, calculation of geometric characteristics - length, size, circumference, area and volume, interior geometric properties: Frenet-Serret trihedron, curvatures of curves, normal, twist and curvatures of surfaces, tangent trihedron, density vector and interior homogeneity of solids, matrices of transformations and projection methods, visibility algorithms, rendering)
- hardware – computer with a user-friendly graphical interface
- structured database for storing developed models
- graphical output/input device (screen, drawing device - plotter, printer, digitiser)
- application software packages (for processing of digital 3D information, processing of technical documentation, calculation of physical and technical parameters of designed products)
- NC-machine, technical equipment and elaborating tools and instruments

3. Representations of geometric figures

A relatively good software basis exists for 2D and 2,5D geometric modelling of figures in the 3D space, so called fire-frame models and polygonal schemes, or sweep representation. Realistic visualisation and modelling of 3D figures – solids is a relatively new and expanding domain of computer graphics, which is closely connected to the development of the diagnostics techniques in medicine, to the new possibilities of the 3D measuring and digitalisation of geometric characteristics, and to the design of optimal layout of material in processing and shaping products, but also to the modelling of virtual reality. The aim of the latter one is to develop unique and complete system providing a realistic representation of solids. There exist several approaches to describe and represent 3D subsets of the space, geometric models of solids, or solid cells. In the following, there will be presented some of the most widely used representations of solids in CAGD systems.

Wire-frame models

Wire-frame models are sets of lines – line or curve segments, which determine separate edges of the object geometric model. They are usually constructed interactively, similarly to the classical hand constructions in technical drawing. Model is stared as a system of lines. Current systems of modelling enable 3D representation of designed model developed interactively, changing the view of some of the edges. Wire-frame models of 3D objects suffer some disadvantage. The most limiting factor is the absence of preciseness and uniqueness of model. There might exist several interpretations (realisations in the space) of the given model, some of which might be non-existing unreal objects. Another disadvantage is no completeness of the view caused by the missing outline of the figure view, which must be constructed by special constructions available in descriptive geometry.

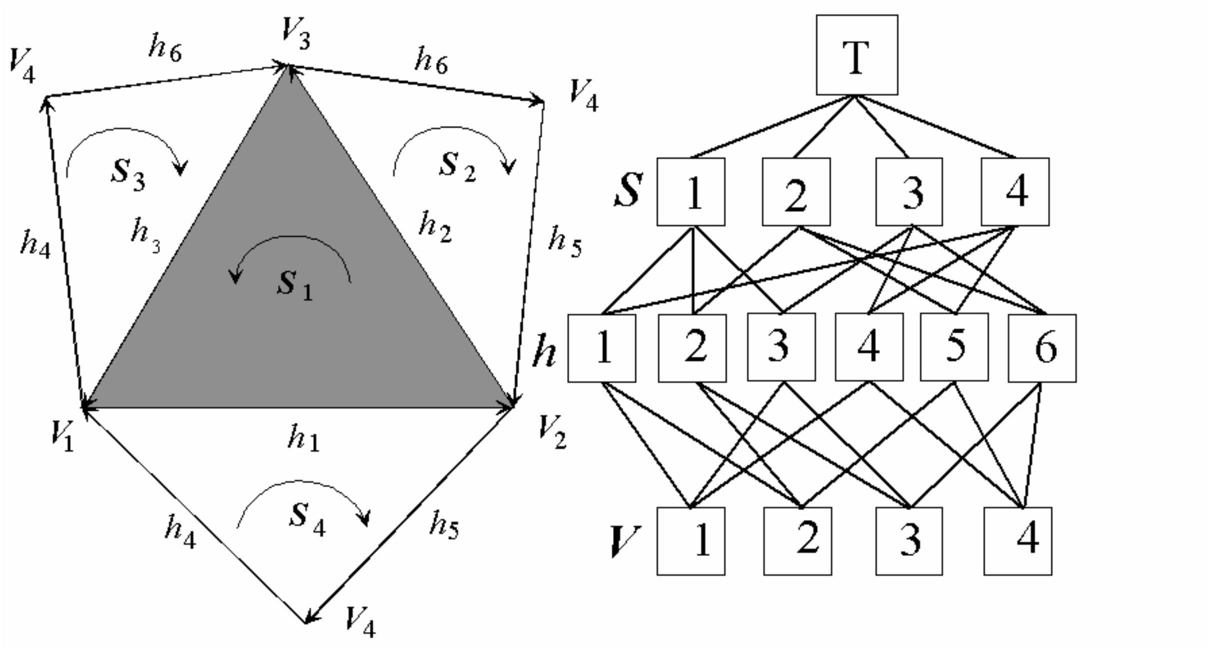

Figure1. Boundary representation of a tetrahedron

Polygonal schemes, boundary B-representation

Polygonal scheme is a direct list of all vertices, edges and faces of a polyhedron, which is a model of 3D object, accompanied by a reference topology of the figure and the incidence structure of the list elements (Fig. 1). This representation was widely used at the development of graphical and visualisation techniques, as: animation, visibility algorithms, rendering. The orientation of the figure boundary is utilised for the calculation of geometric characteristics that describe attributes for visibility and overlapping of figures. Rendering algorithms can be also used to add surface structure, light, reflection and shadow.

Boundary of the solid can be created also by transformation of a curve (or polygon) in the space defining thus a closed surface bounding a volume. In this way the B-representation is linked to the so called sweep representation, which is the predecessor of the parametric representation of figure. Depending on the type of the moving basic figure, there can be modelled geometric figure of different dimensions. In case, that the basic figure is a 2D domain, part of the plane or a surface patch, there can be directly created a general solid, or a solid cell. Animated movement, or deformation of the basic figure, which are both determined as geometric transformations of the space applied to the basic figure, ensure a consequent processing of the mathematical algorithms and this can be regarded as advantage in the interactive regime of modelling, where it is desirable to follow directly the genesis – creation of the designed figure or its deformation.

Parametric models

Model is mathematically represented by a vector (point) parametric function defined on some region and it describes the geometry of object globally. Interior local geometric properties can be represented by derivatives of the parametric function of the figure in terms of differential geometry. Simplicity and universality of this approach, a prompt interactive change of shape, topology and intrinsic geometry, which are implicitly determined by mathematical formulas, make parametric models to represent geometric figures in interactive modelling systems. Disadvantage is in the global description of the object, which implies necessity of the complete data description. Model must be determined by all arbitrary parameters describing its shape and size, which are included in mathematical algorithmic representation as input constants or variables, and therefore form part of the database. Advantage is a fast processing, completeness, uniqueness and correctness of description of the ceated figure and possibility to see the animated process of the model visualisation (Fig. 2).

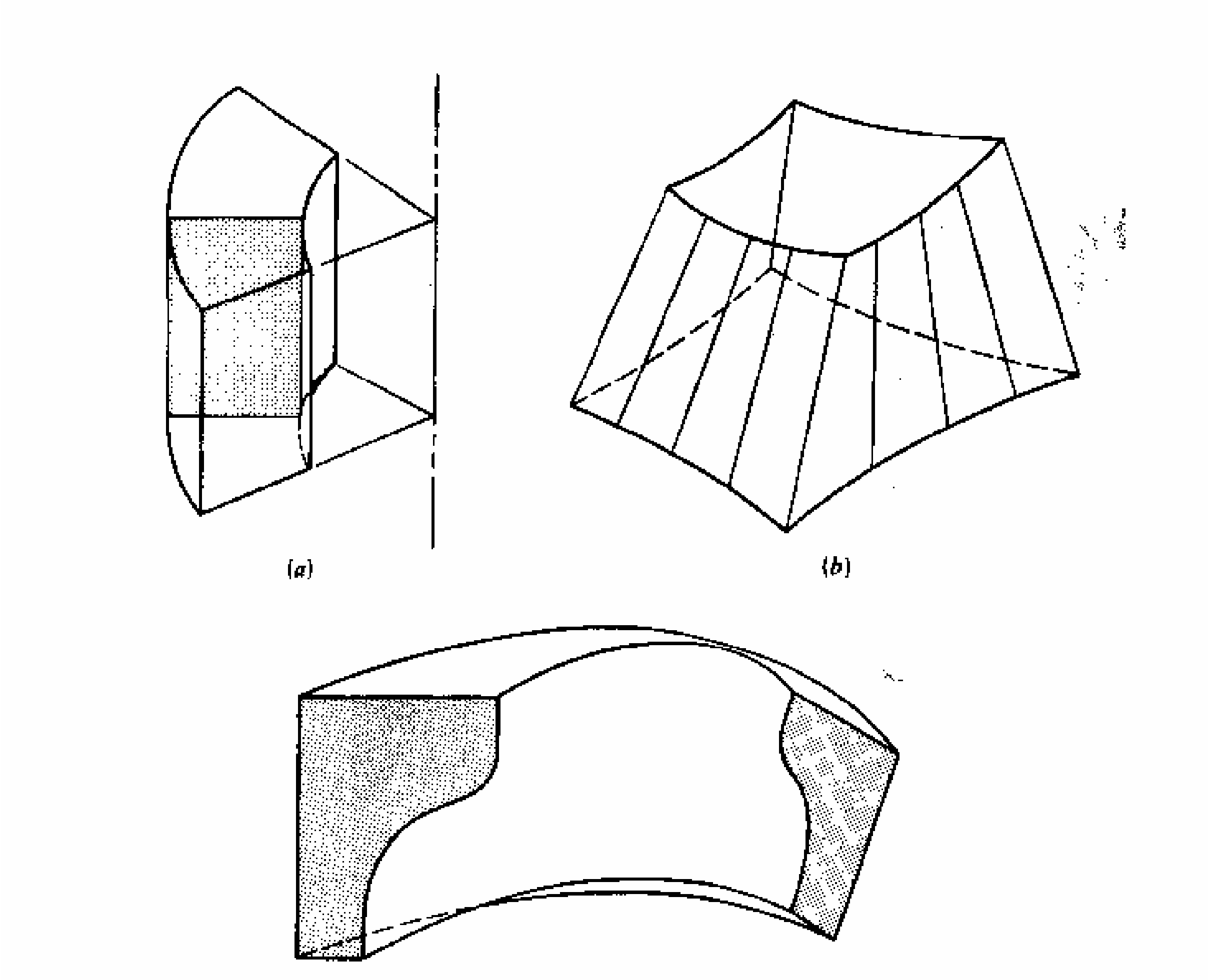

Figure 2. Solid modelled by sweep

Modelling of free-form figures – curves, surfaces, solids

Modelling of surfaces was developed as a substitute for the classical techniques of lofting in design of the parts of aircrafts, cars, and ships in the car and aircraft industry. Today there are commonly used parametric representations of figures (Coons, Ferguson, Bézier, B-splines, β-splines, rational splines, NURBS, Cardinal splines and other curves and surfaces called Free-form figures), which provide a wide range of possibilities to design variety of shapes, forms and positions. These required geometric attributes of the mathematical model are precisely mathematically represented by parametric – vector or point functions of the figures, which approximate precisely enough these characteristics by the choice of the degree and coefficients of the used interpolation polynomials – weight functions. Traditional curves and surfaces had been recently enriched by interpolation solids (hyperpatches, solid cells), intrinsic geometric properties of which were not explicitly determined by means of geometry. Computer graphics played the role of a stimulator for the further development of geometry in this direction. The illustration of the ship hull design and its modification by the change in the basic figure is in Fig. 3.

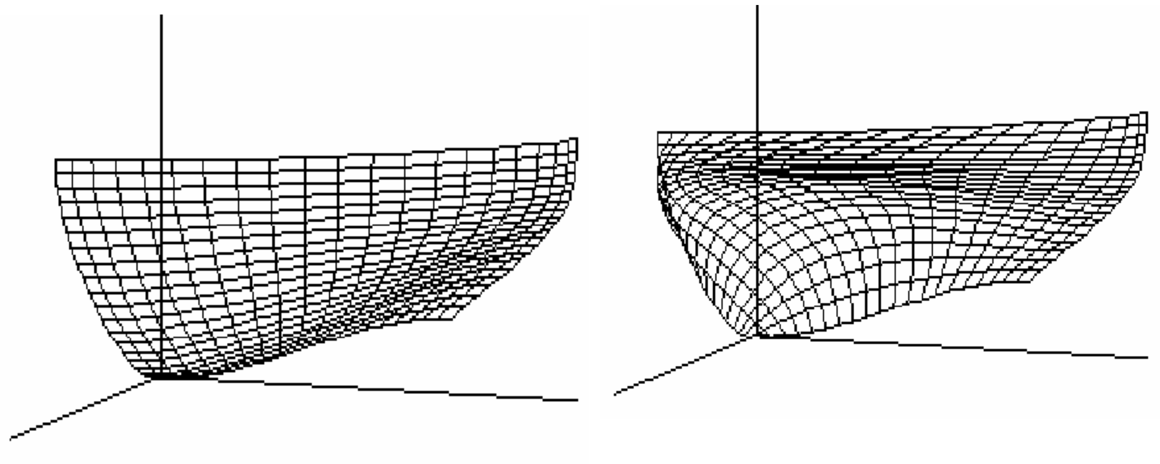

Figure 3. Modified surface patch by the change in the basic figure points

Variational models

Variational models adopt a new approach to developing representations of objects. These are not constrained by some limiting special types of conditions paused on the shape and properties of the modelled object and some parameters controlling the change in shape can be determined interactively, or locally, thanks to the simultaneous modelling and generation of visual object representation due to the parallel solution of equations and processing of algorithms describing the shape and size. Disadvantage is a longer processing time and possibility to create undesirable incorrect configurations of unreal non-existing objects, e.g. dimensioning the machine parts together with the geometric properties of parallelism or perpendicularity of created solid edges that might be leading to the contradictions.

CSG - Constructive Solid Modelling

This method had been developed at the University in Rochester in the USA and it was widespread in various modifications for construction of complex models composed from elementary parts defined first. Method is a part of many graphical systems (AutoCAD, Modelár, Euclid, and others). Model is described by elementary figures called primitives, which are elementary solids as prism, cylinder, sphere, and which can be adjusted, it means, its size, position in the space and form can be determined by the choice of input parameters. Construction of the model from primitives is described by Boolean logical operations: unification, intersection and difference of sets and by Euclidean transformations. Binary tree, illustrated in Fig. 4, with nodes in separate set operations and transformations describes the procedure of figure composition.

Voxel representation

Next from the modelling principles is a decomposition of the space to cells – voxels in the form of elementary solids, e.g. cubes or regular tetrahedrons**,** which are not necessarily equal in size. Modelled object can be divided to regions, which are investigated with respect to the incidence with separate voxels. The largest area of application of this method is in medicine, where object is composed from a sequence of 2D profiles obtained from the digital input device (tomography), positioning them to the basic cube according to the incidence with the separate voxels. Part of the solid in-between separate profiles can be then interpolated. Modelling solid is intrinsically inhomogeneous, it is inscribed to the basic cube, and its intrinsic geometry and properties play the dominating role in investigating special characteristics for the particular applications. The positioning in the basic cube can be easily represented by the octal tree, list of sub-cubes, which are inside the solid. Sub-cubes can be attached attributes - empty, full, heterogeneous. Octal tree can be delivered also linear list.

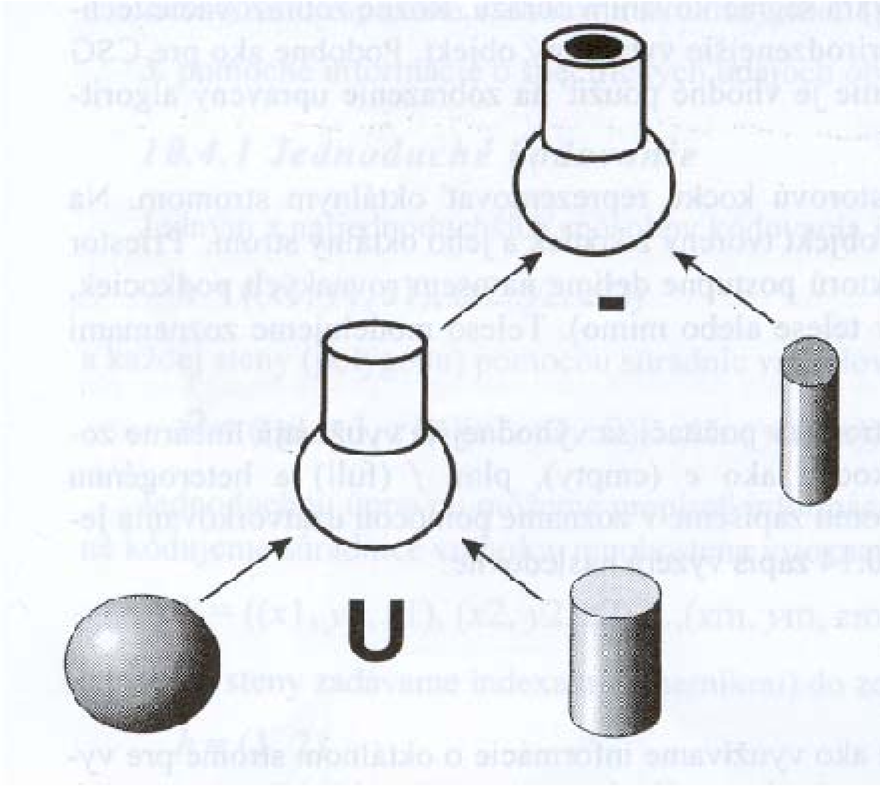

Figure 4. Constructive composition of an object

Sweep - creative representation

Generally speaking, any geometric figure is in computer graphics represented always in both ways, synthetically and also analytically. Relation between synthetic and analytic representation of the modelled figure is expressed by the creative law of the figure. The structure of synthetic representation in the form of an ordered pair (U, g) is given by the type of the basic figure U and applicable generating principle g. There exist several specific types of synthetic representations of geometric figures, which analytic representations in the form of point or vector functions of more variables differentiable on the given region can be obtained as products of multiplication of the analytic representations of concerned basic figures (points, curves, surfaces, solids, animations) and generating principles (geometric transformations, classes of geometric transformations, approximations, interpolations). Geometry of the figure, curvilinear and homogeneous coordinates of separate points, or global and local intrinsic geometric properties of figure in the neighbourhood of the point can be described by methods of differential geometry.

Fractal - fuzzy modelling

Simulation of processes and phenomena, modelling of deterministic non-periodic chaotic behaviour of systems proving unpredictability, strange self-similarity, and sensitivity to initial conditions (butterfly effect) are parts of fractal and fuzzy modelling. In addition to realistic visualisation of natural objects (cost, mountains, plants, animals, veneral system of human organism – two interrelated systems of veins and arteries, lungs – surface of maximal area (equal to the area of a tennis playground) for a minimal volume), the fractal modelling provides the chance to visualise natural forms of uncertain boundaries and of an uncertain shape - fuzzy geometric objects (clouds, flame, smoke, wave motion on the water surface) and chaos (turbulence, winds, whirlpools), as a new form of order. Fuzzy modelling of natural processes and phenomena (qualified weather forecast, oil diffusion through the broken surface, polymerisation process, hair in movement, mimics of the human face, motion of muscles, ageing process – appearing wrinkles and a change of physiognomy) presents the geometry of the Nature.

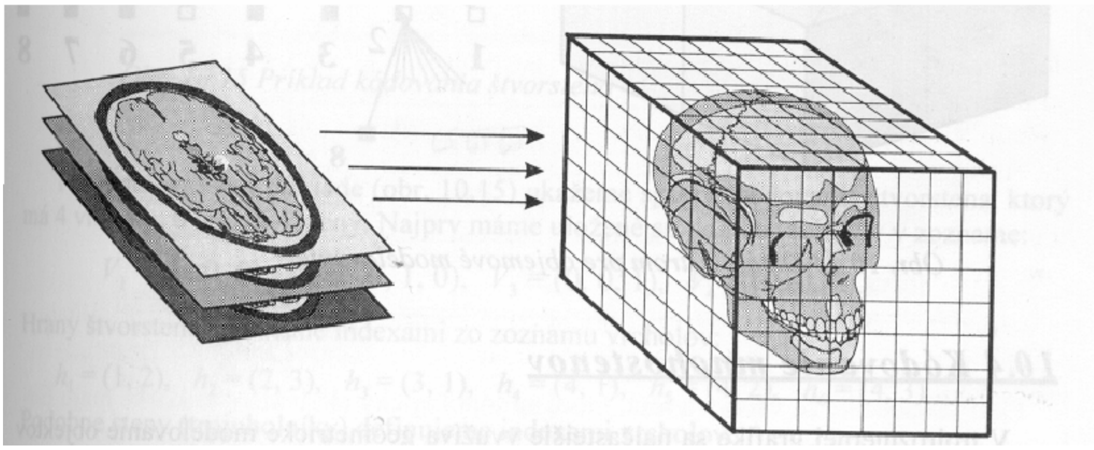

Figure 5. Voxel representation

Free form deformations

One of the latest methods of geometric modelling of space figures are methods of free-form deformations, non-linear transformations of space.

Grid of $(m + 1) \times (n + 1) \times (l + 1)$ points in the space

$$P_{ijk},\ 0 \le i \le m,\ 0 \le j \le n,\ 0 \le k \le l$$

defines the basic space, where a Bèzier solid cell is determined on the region $\Omega = [0,1]^3$ by the differentiable point function of three variables

$$\mathbf{r}(u,\mathrm{v},w) \ = \ \sum_{i=0}^{m}\binom{m}{i}u^i(1-u)^{m-1}\sum_{j=0}^{n}\binom{n}{j}v^j(1-v)^{n-j}\sum_{k=0}^{l}\binom{l}{k}w^k(1-w)^{l-k}P_{ijk}$$

Interior curvilinear coordinate system of solid is $(X_0, \mathbf{U}, \mathbf{V}, \mathbf{W})$.

Created – deformed figure is inserted into the grid (Fig. 6), while curvilinear coordinates (u,v,w) of any point X of the figure can be determined in the form $X = P_{000} + u\mathbf{U} + v\mathbf{V} + w\mathbf{W}$.

Deformation of the space is defined by the change in the position of the vertices P_{ijk} in the basic grid and by creating a new grid determined from points Q_{ijk}, while curvilinear coordinates of point X_{def} after deformation are given by formula

$$X_{def} = r_{def}\,(u,v,w) = \sum_{i=0}^{m}\binom{m}{i}u^i(1-u)^{m-1}\sum_{j=0}^{n}\binom{n}{j}v^j(1-v)^{n-j}\sum_{k=0}^{l}\binom{l}{k}w^k(1-w)^{l-k}P_{ijk}$$

Setting the initial conditions determining the type of the deformation (invariant some of the geometric characteristics - area, volume, boundary, and others.), specific deformations can be defined, e.g. interior deformation, volumetric, and so on.

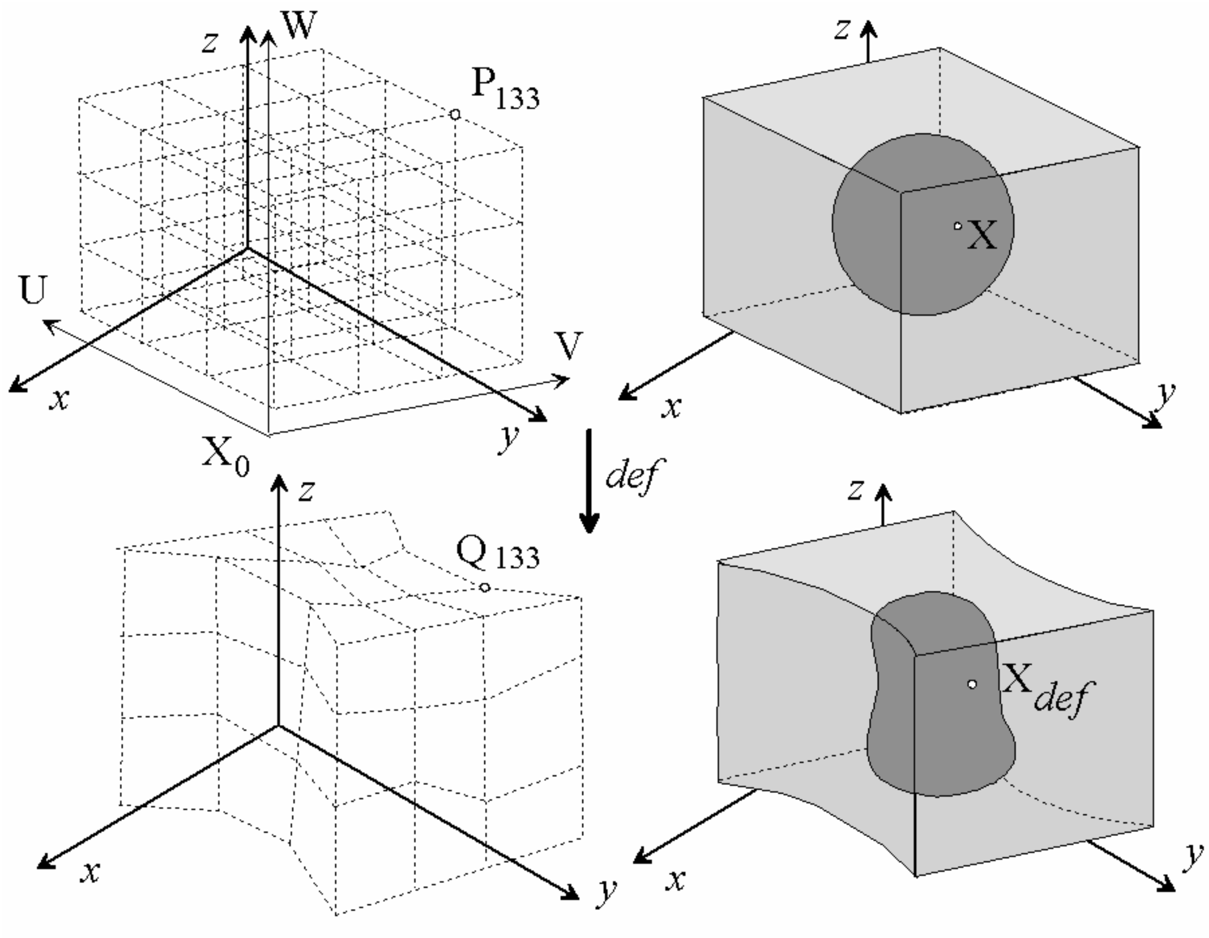

Figure 6. Free-form deformation

To define the basic space the Bèzier grid of tetrahedrons can be used with the vertices in points P_{ijkl} . Barycentric coordinates of the basic tetrahedron points are then

$$P(r, s, t, u) = \sum_{i+j+k+l} P_{ijkl} B^n_{ijkl}(r,s,t,u)$$

$$B^n_{ijkl}(r,s,t,u) = \frac{n!}{i!\,j!\,k!\,l!} r^i s^j t^k u^l,\ r+s+t+u=1,\ i+j+k+l=n$$

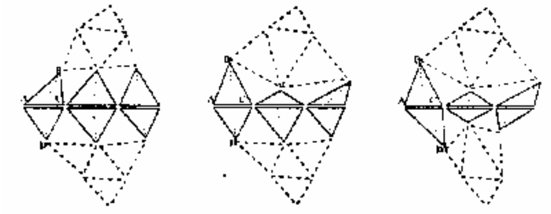

Figure 7. Tetrahedral grid modification

Modifying the position of the tetrahedron vertices in the grid (Fig. 7), a local deformation of the space is defined, to which also the inserted figure is subdued.

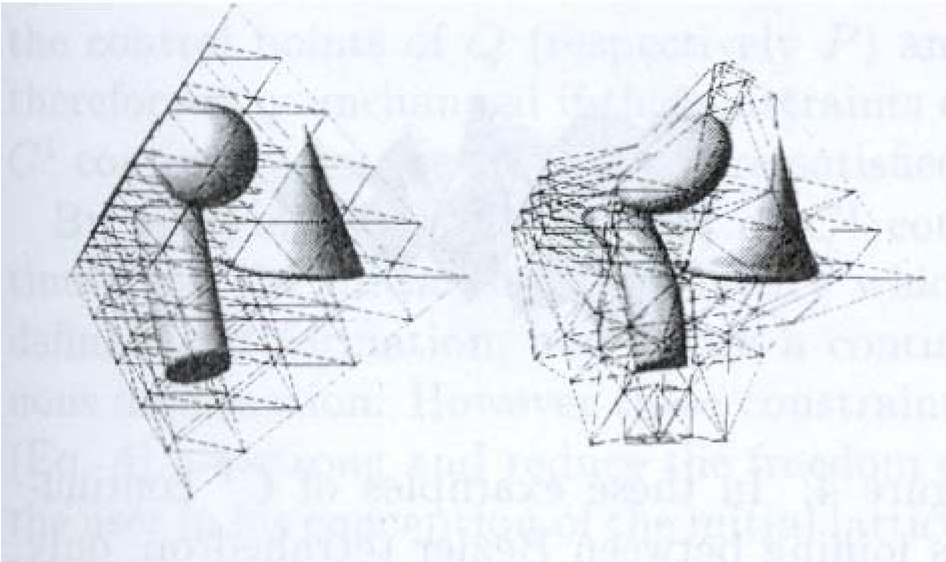

Figure 8. Free-form defomations of elementary solids

4. Conclusions

Computer modelling today is a common practise in industry, in technical applications, and also in theoretical scientific activity. It is often used for simulation of special scientific and technical processes and for visualisation of models of data structures in many scientific fields. Modelled objects are more and more complex graphical figures representing objects from the real world. Analytic and synthetic representation of geometric figures are two possibilities of the formalised description of geometric figures, which are in classical theories strictly distinguished, z usually not in the favour of the synthetic approach. Synthetic approach is anyhow, dominant with respect to the modelling process, for which geometry and geometric attributes of model have a prior meaning, not mathematical formulas, which can be received as the result of synthetic creative laws.

References

[1] Foley, J.D., van Dam, A., Feiner, S., Hughes, J., Philips, R.L.: Introduction to Computer Graphics, Addison-Wesley, 1993

[2] Mantyla, M.: An Introduction to Solid Modeling, Computer Science Press, 1988

[3] Medek, V., Zámožík, J.: Osobný počítač a geometria, Alfa Bratislava, 1991

[4] Mortenson, M.E.: Geometric Modeling, John Wiley & Sons, New York, 1985

[5] Preparata, F., Shamos, M.I.: Computational Geometry, Springer - Verlag, 1985

[6] Quilin, D., Davies, B. J.: Surface Engineering Geometry for Computer-aided Design and Manufacture, Ellis Horwood Limited, Chichester, 1987

[7] Ružický, E., Ferko, A.: Počítačová grafika a spracovanie obrazu, Sapientia 1995,

[8] Velichová D.: A New Formalism in Teaching Geometry: Creative Geometry, Proceedings of COMPUGRAPHICS´96, Paris, France, 1996, pp. 293-297

[9] Velichová, D.: Interpolation of Solids, Proceedings of 7th ICECGDG, Cracow 1996, pp. 161-165

[10] Zámožík, J., Surová, A., Vranková, E.: Základy počítačovej grafiky, Vydavateľstvo STU Bratislava, 1993