Heat transfer and air flow in buildings *

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Abstract

Modelling of heat propagation in buildings, consisting of rooms separated by inner walls and covered by outer walls, roofs, etc., belong to serious non-trivial problems of civil engineering design. Especially in rooms the classical heat conduction analysis (using the Fourier - Kirchhoff approach) gives no reasonable results because the air flow, driven by outer climate quasiperiodic changes, forces quite other time-variable redistributions of a temperature field. Thus, three physical conservation principles have to be respected – of mass (generating one continuity equation), of momentum (generating three equations of viscous air flow of the Navier - Stokes type) and of energy (generating one equation of heat transfer that can be identified with certain extended version of the Fourier - Kirchhoff equation). Nevertheless, such general system is complicated and corresponding numerical calculations are slow and expensive. This paper discusses admissible simplifications and presents the original MATLAB-based software code for the analysis of simultaneous air flow and heat conduction both in rooms and in construction parts, implementable on standard personal computers without higher hardware of software requirements. One two-dimensional example with a realistic room size and air characteristics is demonstrated.

Key words: building design, heat transfer, air flow, Fourier - Kirchhoff equation, Navier - Stokes equations, Boussinesq approximation, method of Rothe, finite element method, MATLAB-based simulation.

1 Introduction

For all rooms of buildings and other parts of structures where long-time human activities are assumed the good prediction of a temperature field development, caused by outer climatic changes, artificial heating, air conditioning, etc., is necessary. Simple calculation methods, based on the analysis of the classical heat conduction equation, are friendly even to persons with a non-positive relation to higher mathematics, but they are far to the reliable description of reality because the most part of volume of such buildings and similar structures consists of air in rooms with a very low thermal conduction factor, but with a good ability of flow through such empty space. This is a sufficiently serious motivation for the development of physically and technically reasonable (not only mathematically correct) analysis of such temperature redistribution; this analysis (unlike the complete

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study of the system of Navier - Stokes equations) should not require large and expensive calculations, making use of parallel computer architectures.

The aim of this contribution is to show (at some non-trivial level of understanding the heat transfer in buildings) that this process of changes of a temperature field, determined by air flow (and usually only slightly modified by classical heat transfer) can be simulated using the standard personal computer with (or without) installed MATLAB software. All software modules are designed to be convertible to modules included into C++, Fortran, Pascal and similar programs. In this way they are able to create elements of a more complex computational system for the thermal (and other, respectively) analysis of large structures and constructions in civil engineering.

2 Methods of modelling of thermal transfer in buildings

For the study of heat transfer through all layers of walls, roofs, etc., the application of the classical equation of heat conduction can be successful. In the most simple case of the one-dimensional heat conduction we have one linear ordinary differential equation of the second order with a variable temperature field (this is the famous historical Fourier - Kirchhoff equation – see [6], p. 193) that can be completed by a couple of additional transfer conditions and (at least for a stationary version with constant material characteristics) easily solvable, using elementary knowledge of the analysis of differential equation. In such access we ignore the fact that real material characteristics are typically (usually not very strongly) temperature-dependent (this disturbs the announced linearity); moreover, it is very important whether the applied (typically porous) material contains pores occupied by air with vapor, (condensed) liquid water or ice. Especially the last two cases are very unpleasant (but not able to be avoided completely) in practice and their modelling is much more complicated; the more detailed information (including a lot of references), concerning complex problems of "HAM (= heat, air and moisture) modelling" in building materials can be found in [10] a [13].

The introduced simple approach is not strictly limited to one-dimensional problems: natural generalizations to two- and three-dimensional problems are available, but the resulting equation is partial and corresponding transfer conditions must be applied on the whole boundary of a domain of interest. From both thermal insulation and accumulation reasons most constructions in buildings are layered (with jump discontinuities in material characteristics); therefore the reformulation of an original classical differential equation to some weak or variational form (friendly to finite element techniques) can be recommended.

Nevertheless, this approach (both in a differential and in an integral form) is a failure in case of simulation of heat transfer through rooms, containing air. The low heat conduction factor explains the fact that most predicted (very slow) temperature redistributions are bad, in contradiction with practical experience and all quantitative measurements: in fact, in this way we make the proper analysis of a nearly insignificant process, but the essential changes in a temperature field are caused by (much faster) motion of air particles. This discouraging conclusion can be compensated by various ways – here we shall mention four representative ones:

- the mathematically simple, but not very apposite idealization of a temperature field as constant in the whole room, variable only in time, depending on the heat transfer through walls – such system approach is presented in [11] and [12],
- the formulation of a coupled evolution problem of heat transfer and fluid (air) dynamics with a low (but finite) viscosity, based on some generalized version of the system of Navier - Stokes equations, probably with large and slow calculations, crashing in case of insufficiently robust software support (for example, in the numerical analysis of large systems of nonlinear algebraic equations as discretized versions of partial problems of elliptic type, needed in all time steps for the construction of Rothe sequences in the method of discretization in time),
- the aspiration to simplify the preceding formulation, based on the neglecting of non-substantial terms, using the proper physical and technical analysis, leading to some relatively cheap and sufficiently reliable credible simulation of reality,
- the use of commercial software packages of different extent, expertness and prize (as ANSYS, SYSTUS, COSMOS, I-DEAS, FLOTRAN, etc.) for the solution of problems of continuum mechanics as "black boxes" (the incompleteness of theoretical manuals and various complications in all attempts to incorporate needed functions into some software system with quite other professional orientation and priorities can be expected, apart from the license difficulties), alternatively alienation of (at least seemingly convenient) software means at some "friendly" institution.

In the rest of this paper we will follow the third approach. For the sake of brevity of notation and simplicity of explanation we shall concentrate our attention to a partial problem of a temperature development in one typical room, occupied by air; this problem is the most complicated both from the point of view of mathematical formulation and from the point of view of an optimal design of numerical algorithms and their implementation.

3 Mathematical analysis of air flow and heat transfer in a room

In the analysis of air flow and heat propagation in an arbitrary room, described by a domain Ω geometrically, in general in the three-dimensional Euclidean space, the knowledge of the following air characteristics on Ω and $\partial\Omega$ will be needed:

λ heat conduction factor [W/(m K)] on Ω (cf. [6], p. 57),

α heat convection factor [W/(m^2 K)] on $\partial\Omega$ (cf. [6], p. 56),

c thermal capacity [J/(kg K)] on Ω (cf. [6], p. 52),

η viscosity factor [Pa s] on Ω (cf. [5], p. 223) for the ideal viscous (Newton) relation between tangential stress and strain rate for certain air volume.

These characteristics are (not very strongly) temperature-dependent. Corresponding functions of temperature can be set using [2]; in practical calculations for usual temperature range, required for human activities in rooms by technical standards, the approximation of data from corresponding tables, using the cubic polynomials generated by the least square method, seems to give good results. For the rough orientation the values

$$\lambda \approx 0.025\,\mathrm{W/(m\,K)}\,, \quad \alpha \approx 7\,\mathrm{W/(m^2\,K)}\,, \quad c \approx 1010\,\mathrm{J/(kg\,K)}\,, \quad \eta \approx 18 \times 10^{-6}\,\mathrm{Pa\,s}$$

are applicable, too, together with the usual air density $\rho = 1.293\,\mathrm{kg/m^3}$ (cf. [5], p. 225).

In [13], p. 39, from the conservation principles for mass, inertia and energy the general system of equations of air flow in a room, caused by temperature changes, has been derived. This system is rather complicated and the suggested general algorithm cannot be implemented on a standard personal computer completely. Although some turbulent flow can be expected, a lot of terms in presented equations may be simplified of neglected at all. Especially the Mach number Ma (a typical flow rate, divided by a rate of sound propagation in air) can be supposed to have the property $\mathrm{Ma} \ll 0,3$; in this case [3], p. 2, recommends to ignore the air compressibility in the continuity equation, obtained from the mass conservation principle. This leads to the so-called Boussinesq approximation (explained in [3], p. 915, in details), making use of an additional auxiliary air characteristic – of the extensibility coefficient γ; here we shall have $\gamma = 0.00366\,\mathrm{K^{-1}}$ (its constant value corresponds to an isothermal process).

For the preliminary qualitative analysis of air flow with heat propagation some dimensionless technical constants are interesting; a very extensive overview of such constants can be found e. g. in [6]. In this paper we shall need the Prandtl number Pr (by [6], p. 89), the Rayleigh number Ra (by [3], p. 214), and a the Reynolds number Re (by [5], p. 229), from definitions

$$\mathrm{Pr} = \frac{\eta c}{\lambda}\,, \qquad \mathrm{Ra} = \frac{\rho^2 g \gamma \overline{T} H^3}{\eta^2} \cdot \mathrm{Pr}\,, \qquad \mathrm{Re} = \frac{\rho v H}{\eta}\,.$$

In such definitions we do not know some quantities exactly, but we are able to do their reasonable estimate: H should be a characteristic length that can be taken as the height of our typical room $H = 5\,\mathrm{m}$, and $\overline{T}$ has to be an expected difference between the maximum and minimum temperature – for this purpose we are allowed to set $\overline{T} \approx 2\,\mathrm{K}$. In this way we can evaluate $\mathrm{Pr} \approx 7.272$ and consequently $\mathrm{Ra} \approx 3.368 \times 10^{10}$. A reliable estimate of the air flow rate v (in absolute value), needed for the evaluation of Re, is not available, but we can come out from the discussion in [3], p. 372: the influence of gravity effect is negligible for $\mathrm{Re} \gg 10^4\,\mathrm{Ra}$. This yields

$$v \gg \frac{10^4\,\mathrm{Ra}\,\eta}{\rho H} \approx 9.378 \times 10^8\,\mathrm{m/s}\,.$$

To reach such velocity is quite impossible in practice (the maximal velocity of the hurricane Ivan in Florida in September 2004 was lesser than 50 m/s).

Thus, an announced idea that the main physical process in our model must be the air flow in a room, conditioned by the presence of gravity and caused by some temperature change in outer environment with the secondary temperature redistribution in a room, seems to be correct; the effect of the classical heat conduction (especially with respect to the low value of λ) will be less important. Nevertheless, drastic differences in comparison with "nice textbook" character of viscous (turbulent) flow modelled in [3], p. 215, with $\mathrm{Pr} = 10^{-1}$ and $\mathrm{Ra} = 10^5$ have to be expected; substantial complications both in mathematical analysis and in practical computations will be mentioned later.

These considerations validate several simplification of equations derived in [13], pp. 39-41. Nevertheless, following these equations arbitrary indices $i, j \in \{1, 2, 3\}$ will be taken as sum indices in sense of the Einstein summation rule, and a dot will denote a partial time derivative by a variable x_i or x_j where $x = (x_1, x_2, x_3)$ are Cartesian coordinates of an arbitrary point of Ω, or on $\partial\Omega$, respectively (if two indices occur, a second partial derivative by corresponding variables).

Following both [8] and [1] (in agreement with [3], pp. 15 and 158), let us treat the air density as constant in convective and non-stationary terms and as variable only in gravitational terms. Then it is sufficient to work with two independent fields of time-variable quantities on Ω: of the temperature T and of the (three-component) flow rate $v = (v_1, v_2, v_3)$. This is possible because the differential continuity equation, coming from the mass conservation principle, gets a very simple form

$$v_{j,j} = 0\,, \tag{1}$$

restricting the choice of admissible flow rates on Ω only; the variable pressure from the Gay - Lussac law and also the variable density can be then eliminated (using the method described in [3], p. 9 – the reference pressure corresponding to the standard air pressure in the corresponding height is considered). The inertia conservation for certain reference temperature T_c (influencing the setting of the coefficient γ) yields a trinity of differential equations of air flow on Ω

$$\rho\dot{v}_i + \rho v_{i,j} v_j - \eta\varepsilon_{ij,j}(v) = -\gamma g_i (T - T_c)\,; \tag{2}$$

here $g_1 = g_2 = 0$ a $g_3 = -g$ and on $\partial\Omega$ homogeneous Dirichlet boundary conditions $v_1 = v_2 = v_3 = 0$ are preserved. The energy conservation then implies (if no additional energy sources are installed in a room) one differential equation of heat conduction on Ω

$$\rho(cT)\dot{} + \rho(cT)_{,j} v_j - \lambda T_{,jj} = -\eta\varepsilon_{ij}(v)\varepsilon_{ij}(v)\,, \tag{3}$$

supplied by some boundary condition of heat convection on $\partial\Omega$ in form

$$\lambda T_{,j}\nu_j + \alpha(T - T_*) = 0 \tag{4}$$

where T_* represents the prescribed development of the outer temperature in time a $\nu = (\nu_1, \nu_2, \nu_3)$ denotes unit vectors of outer normals to $\partial\Omega$.

The transformation of the system of equations (1), (2), (3) and (4) to a weak form, based on the Green - Ostrogradskiĭ theorem (on integration by parts), can be done in the same way as in [13], p. 44: the boundary condition (4) is included in the unique integral equation of heat conduction, coming from (3), and the integral equation of air flow, generated by (2), has to respect the condition (1) for every choice of test air flow rates. The resulting equations can be also divided by the density ρ and by some reference value of the specific heat c (related technical difficulties can be overcome using the same access as in [13], p. 42).

Let us now notice what happens if we try to ignore the air viscosity, and therefore to set $\eta \approx 0$. For simplicity let us discuss only a stationary problem (with $\dot{v}_i = 0$) and believe that we are able to know T in all equations (2) in advance. We obtain three differential equations of the first order; consequently, we cannot expect strict preservation of zero velocities on $\partial\Omega$ (this is most evident in case of an one-dimensional problem where we would receive one strongly nonlinear equation of the first order, but with two Dirichlet boundary conditions) and, moreover, the preservation of one additional condition (1). For sufficiently viscous liquids the term $-\eta\varepsilon_{ij,j}(v)$ dominates in (2); such term is only slightly nonlinear (because the factor η is a function of T); unfortunately, for air this is not true – here the dangerous nonlinear term $\rho v_{i,j} v_j$ prevails. In practical calculations this fact is able to force always the analysis of the whole problem as evolution problems (the stationary problem is closed to a problem which is not solvable – consequently after the discretization no appropriate solver for the corresponding large system of strongly nonlinear algebraic equations may be available). Even the introduction of (Lebesgue and Sobolev) function and abstract function spaces and subspaces containing sought solutions is not easy; for more information we can refer to [4] where some theoretical (rather complicated) formulae, estimating errors of approximate solutions in norms of such spaces, are derived, too.

Although unpleasant nonlinearities occur also in the equation (3), the main source of computational complications stay facts discussed in the previous paragraph. The direct solution of a stationary problem is often impossible (if needed, e. g. in the description of some initial status, an iteration procedure from an artificial formulation of an evolution problem would be applied). Nevertheless, the following numerical results will demonstrate that the prediction of heat propagation, conditioned by air flow in a room, is possible (at least for the two-dimensional problem) with help of usual personal computer and standard MATLAB software.

The evolution problem, based on the weak formulation of (2) a (3) with the incorporated condition (4), can be studied using the construction of Rothe sequences of approximate solutions (solutions of partial problems in time steps), corresponding to some semidiscretization in time (cf. [13], p. 43). It remains to force the validity of (1) in every time; this can be done using the algorithms of type SIMPLE, analyzed in [15]. Such partial problems can be discretized on a three-dimensional (or on a simplified two-dimensional) domain Ω using the finite element technique (alternative ways of discretization, making use e. g. of the finite volume or difference method, we will not discuss here). In this paper we shall present only one easily intelligible two-dimensional problem – from the geometrical point of view it may be characterized as a vertical cut of one (sufficiently long) room.

4 MATLAB-based software development

The software package MATLAB and its incorporated toolbox PDE (for the numerical analysis of partial differential equations and their systems) offer functions for triangular mesh generation on an arbitrary two-dimensional domain, functions for allocation and analysis of sparse matrices, generated by discretized partial differential operators and (implicitly included) functions for the effective analysis of large sparse systems of linear algebraic equations. For the solution of nonlinear algebraic systems numerous functions from the mathematical library NAG are available. MATLAB offers also tho possibility to call functions written in its own language (in certain standard form) from codes written in common programming languages, namely from C++, Pascal and Fortran (therefore program debugging can apply achievements of such environments as Visual C++, C++ Builder or Delphi). Let us notice that calling MATLAB functions can be avoided elegantly, using the method described in v [14], but the amount of programmer's work would increase in our case significantly (but, from the juridical point of view, no MATLAB license would be necessary).

For the numerical analysis of the system (2) (including (4)) a (3) the original program code in the MATLAB language has been created; the preservation of (1) is forced in this code by the SIMPLE algorithm (with matrix pseudoinversions, applying MATLAB functions again). The setting of several basic inputs is interactive, other data are read from common text files. The graphical outputs of computation results make use of the MATLAB postprocessing; outputs are alternatively saved in files in the ∗.epa format (which means "encapsulated PostScript") for future interpretation (e. g. in this contribution). Special original additional functions support various presentation forms in discrete time steps or animation on selected time intervals both of the temperature T and of the air flow rate $v = (v_1, v_2)$. The call of this software from external programs is assumed using the MATLAB Engine; this software access is introduced in Part 5 of the MATLAB documentation [7].

5 Numerical results

As our model problem let us study a vertical cut through one room with the same width and height $H = 5\,\mathrm{m}$ now. (The height may seem to be large, but all participants of the 3^{th} mathematical workshop can see that the height of rooms at the Faculty of Civil Engineering in Brno in frequently higher.) The outer temperature oscillates quasiperiodically in day and year cycles; in our example we shall trace only one relatively quick (but realistic) temperature change: the constant temperature $19^\circ\,\mathrm{C}$ in our whole room will start to increase in time $t = 0\,\mathrm{s}$ on the outer wall surface such that in time $t = 400\,\mathrm{s}$ it will reach $21^\circ\,\mathrm{C}$, but on the inner wall surface it will be unchanged and on the floor and ceiling its distribution will be linear. The temperature change will be continuous in time; for its description we shall use the elementary function $\cos(\ldots)$. The square domain (every side is $5\,\mathrm{m}$ long) we shall divide (rather roughly) to identical triangles using the regular mesh whose geometry is clear from Fig. 1. Using such mesh, in addition to the distributions of

T a v we can see some purely numerical effects (with no physical background), caused by different numbers of triangle edges going to various mesh nodes.

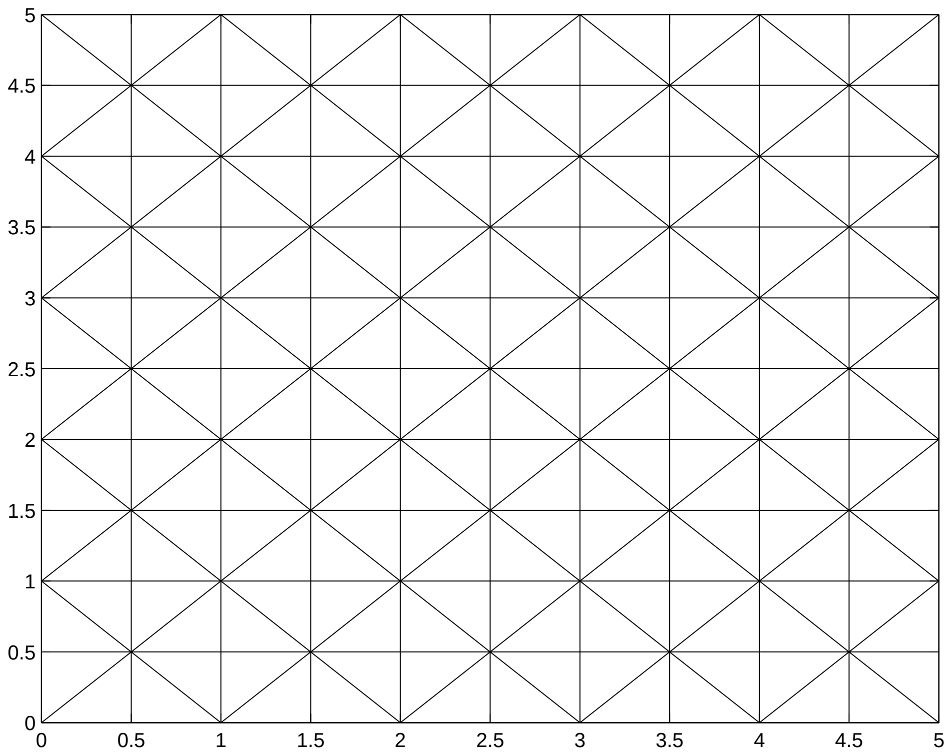

Fig. 1: Regular decomposition of Ω to triangular finite elements

The figure couples, presented on next pages, depict (each one from a couple in an other way, as T is a scalar field, but v is a vector field) the temperature and air flow rate distributions in the following times:

- $t = 200\,\mathrm{s}$ (in one half of the outer temperature enhancement) – Fig. 2 and 3,
- $t = 400\,\mathrm{s}$ (at the end of the outer temperature enhancement) – Fig. 4 and 5,
- $t = 2000\,\mathrm{s}$ (at the end of calculation, far from the end of the outer temperature enhancement) – Fig. 6 and 7.

We can see that the air flow is rather weak (its rate is low) and its assessment is limited to the vicinity of the outer wall. But, even in this case, the temperature redistribution is driven by the air flow; the contribution of the classical hat conduction in nearly invisible in all graphs. Nevertheless, let us remark that we must believe that the air flow has no other sources – we force the zero flow rate on $\partial\Omega$ which can be violated e. g. in case of an open window. From our sequence of figures we can also see that the rough simplified approach of [11] a [12] with the non-variable temperature in a room (in a fixed time) in not a nonsense totally – except some air layer near both walls, the floor and the ceiling (namely near the outer wall) this evidently non-physical assumption seems to be approximately respected in all studied times.

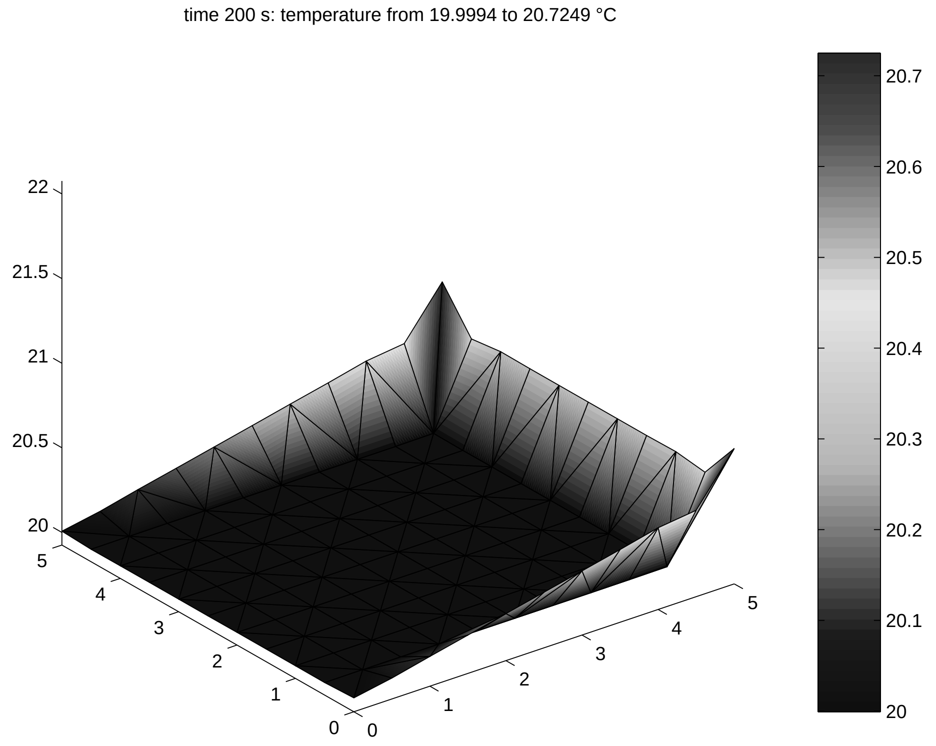

Fig. 2: Temperature in time $t = 200\,\mathrm{s}$

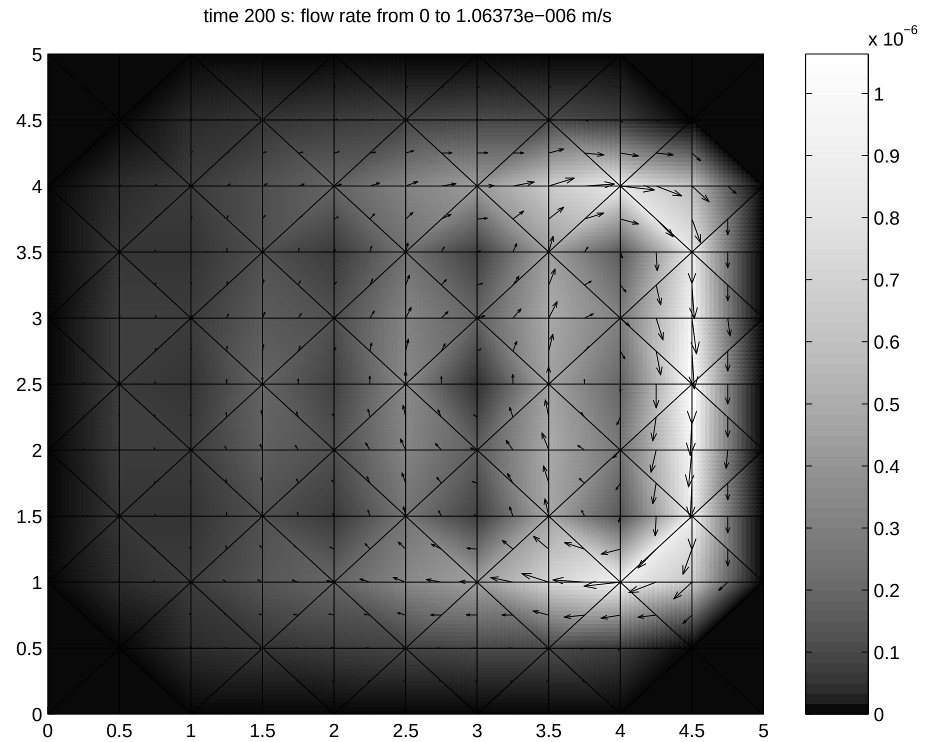

Fig. 3: Air flow in time $t = 200\,\mathrm{s}$

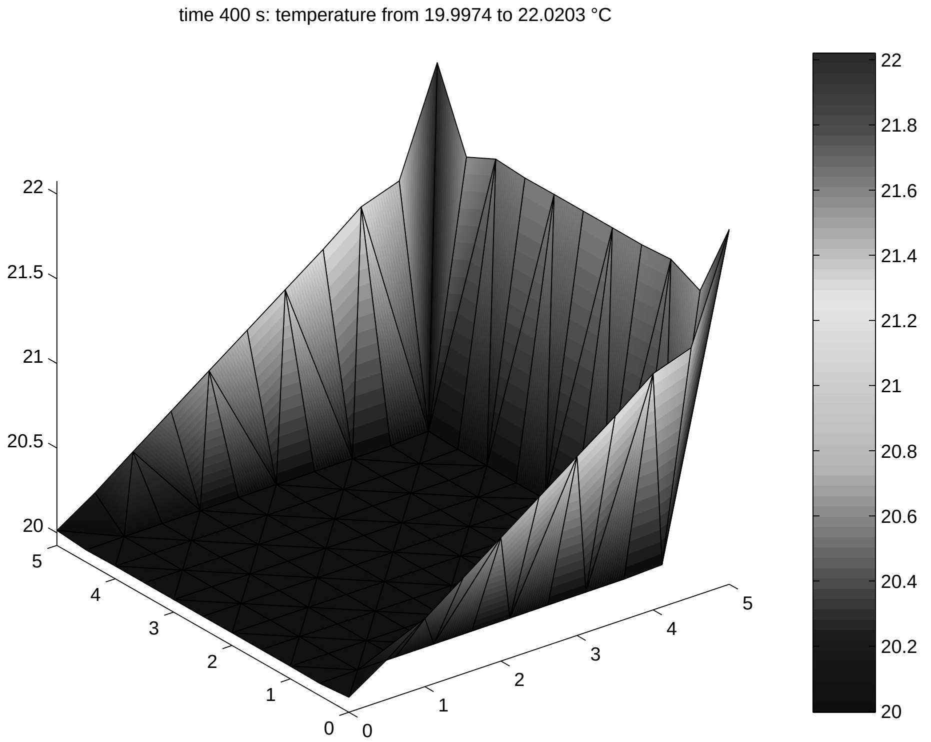

Fig. 4: Temperature in time $t = 400\,\mathrm{s}$

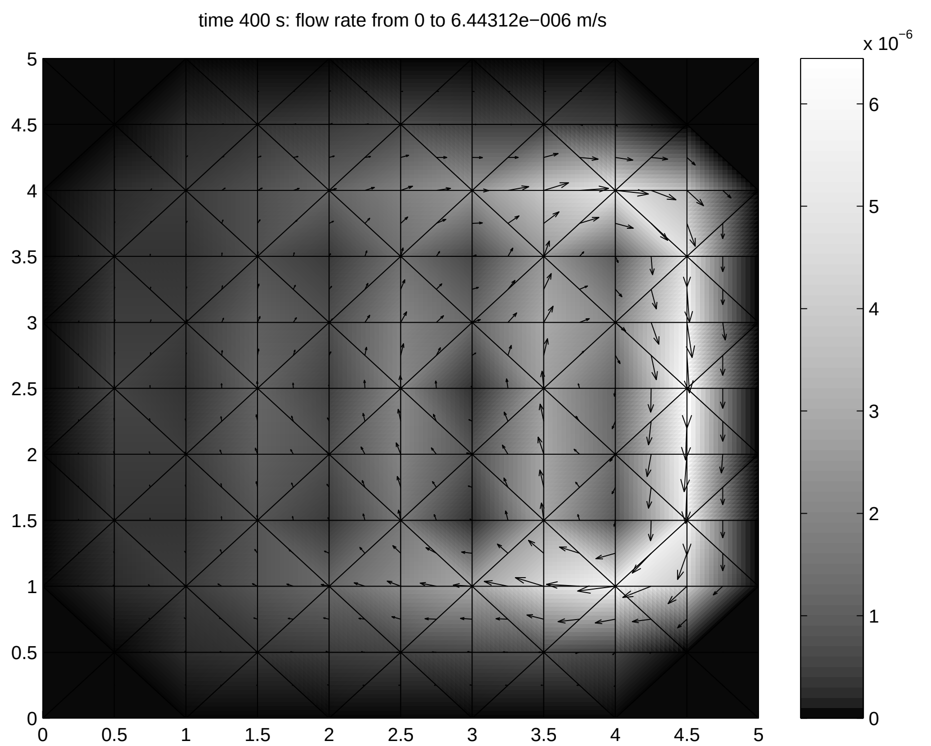

Fig. 5: Air flow in time $t = 400\,\mathrm{s}$

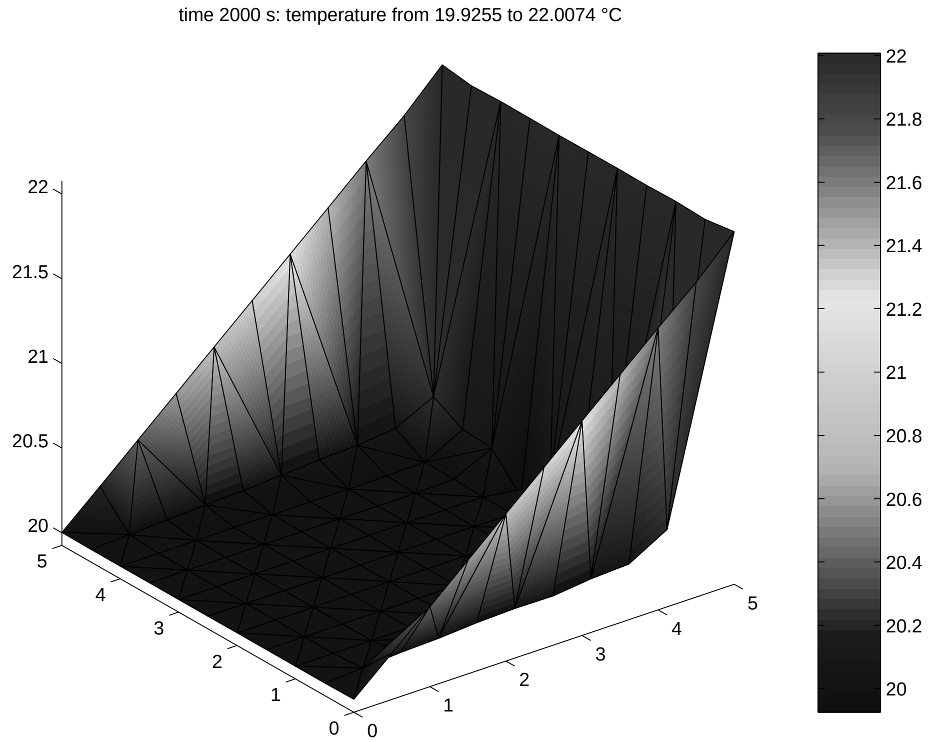

Fig. 6: Temperature in time $t = 2000\,\mathrm{s}$

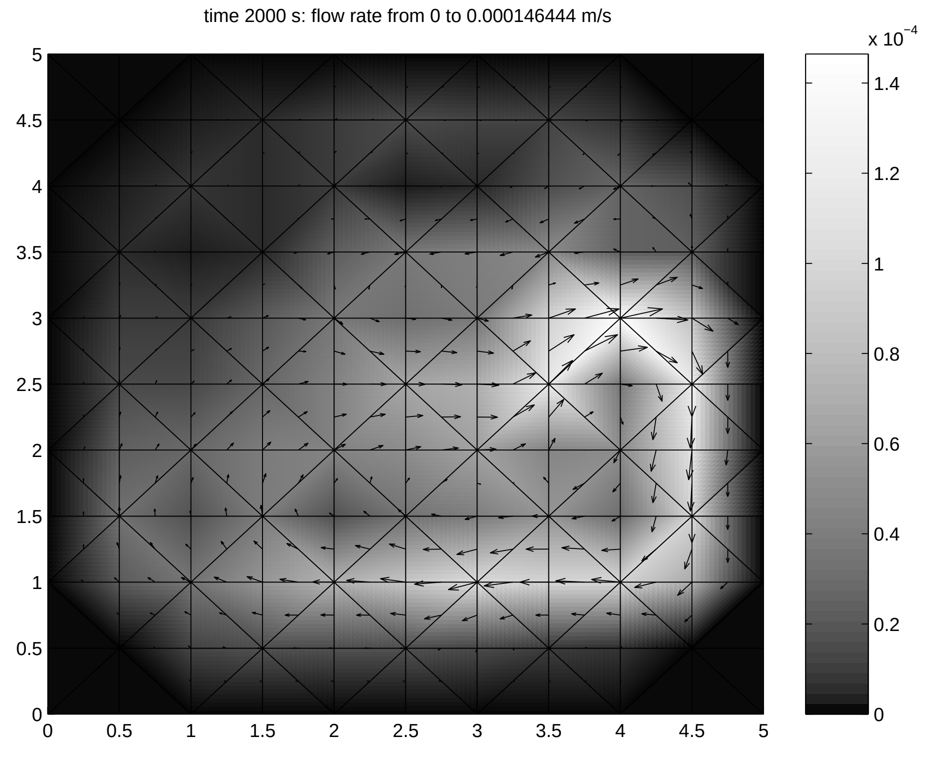

Fig. 7: Air flow in time $t = 2000\,\mathrm{s}$

The resulting air flow character differs from demonstrations from [3], pp. 214-215 strongly. We have accented that we have quite other numbers Pr a Ra. Further numerical experiments (not presented in details in this paper) show that more similar results to above mentioned ones (with one nice great whirl in a square) can be attached with much larger η or with much more λ, respectively (then the distribution of T tends to linear one). Unfortunately, such tricks are in contradiction with realistic air characteristics (functions of T), whose overview can be found in [2].

6 Conclusions and generalizations

The new software offers a chance to prepare a modular system for the thermal analysis of structures, whose part can be also the reasonable prediction of the temperature development in rooms, compatible with the theory of Navier - Stokes equations, without high software and hardware requirements. The same software can be applied to the heat conduction in porous materials in walls, roofs, etc.; in such cases the heat conduction prevails and the air flow (due to the amount and structure of a pore space) is much less significant. Other complications then occur: in many structures no applied materials are homogeneous and isotropic, the air flow is conditioned by the macro- and microstructure of a pore space that, moreover, determines also the moisture transfer (which is much slower than the heat transfer) in various phases, modifying material characteristics radically, etc.

The applied toolbox PDE from MATLAB contains no special support for the problems of flow of liquids (only for the problems of heat conduction, not sufficient here). Nevertheless, general functions for the analysis of boundary problems of partial differential equations and their systems on a two-dimensional domain Ω are useful; some restrictions are forced by strong nonlinearities. This gap has been overcome by the software package FEMLAB, suggested originally as a competitive alternative of the toolbox PDE (now offered as a MATLAB independent software). Its user-friendly properties are seemingly compensated by lower flexibility; high flexibility (together with numerical efficiency and compatibility with other software means) was always one of main advantages of MATLAB (in contrast to the ANSYS-like software). The development of the toolbox PDE (unlike FEMLAB) has been closed yet; therefore it will be necessary to verify also the possibilities and robustness of FEMLAB functions for the study of problems of type “mass and energy transfer”, for our purpose for the simultaneous processes of air flow and heat propagation in buildings, in the very near future.

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