Differential equations of orthogonal grids

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Introduction

In this contribution we summarize and extend well-known examples of orthogonal trajectories in elementary mathematics. Orthogonality is required and useful property in many technical applications.

Consider the system of curves $F(x, y, c) = 0$ in the plane, where $c \in \mathbf{R}$, and let $\Phi(x, y, y') = 0$ be their differential equation. Hence $\Phi(x, y, -\frac{1}{y'}) = 0$ is the differential equation of their orthogonal trajectories $G(x, y, k) = 0$, where $k \in \mathbf{R}$.

Consider two orthogonal systems of curves in the plane $\varrho = f_1(\varphi, c)$, $\varrho = f_2(\varphi, c)$ in polar coordinates, where $c \in \mathbf{R}$. The condition of orthogonality of these systems is

$$\frac{\mathrm{d}f_1}{\mathrm{d}\varphi} . \frac{\mathrm{d}f_2}{\mathrm{d}\varphi} = -\varrho^2.$$

Consider the autonomous system of differential equations in the plane

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(x(t), y(t)\right), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g\left(x(t), y(t)\right), \tag{1}$$

where f a g are continuous functions in a domain $\Omega \subseteq \mathbf{R}^2$. The solution of the system (1) is $x = x(t)$, $y = y(t)$. The graph of solution of the system (1) is a curve in $\Omega \times \mathbf{R}$. The perpendicular projection of the graph of solution to the domain Ω is a curve given in the parametrical form $x = \varphi(t), \quad y = \psi(t)$, where $t \in J \subseteq \mathbf{R}$, called a trajectory of the system (1).

Solving the system $x' = f(x, y)$, $y' = g(x, y)$ is the same as solving differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$ in the domain $G_1 = \{(x, y) \in \Omega : f(x, y) \neq 0\}$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x,y)}{g(x,y)}$ in the domain $G_2 = \{(x, y) \in \Omega : g(x, y) \neq 0\}$. It can be expressed in a symmetrical form $f(x, y)\mathrm{d}y - g(x, y)\mathrm{d}x = 0$.

Using the complex value $z = x + \mathrm{i}\,y$ we can write the autonomous system $x' = f(x, y)$, $y' = g(x, y)$ like

$$z' = F(z), \tag{2}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}t}$ and $F(z) = f(x, y) + \mathrm{i}\,g(x, y)$ is a complex function.

Consider two autonomous systems $x' = f(x, y)$, $y' = g(x, y)$, and $x' = -g(x, y)$, $y' = f(x, y)$. The trajectories of both autonomous systems create an orthogonal grid in $\Omega \subseteq \mathbf{R}^2$.

If we multiply the function $F = f + \mathrm{i}\,g$ by the imaginary unit i, we obtain the function $\mathrm{i}\,F = -g + \mathrm{i}\,f$.

The theory of stationary points of linear autonomous systems is well-known.

Denote $f_1 = \frac{\partial f(x_0,y_0)}{\partial x}$, $f_2 = \frac{\partial f(x_0,y_0)}{\partial y}$, $g_1 = \frac{\partial g(x_0,y_0)}{\partial x}$, $g_2 = \frac{\partial g(x_0,y_0)}{\partial y}$. Suppose that $f(x, y)$, $g(x, y)$ are continuous functions having continuous partial derivatives of second order in the neighbourhood of the point $[x_0, y_0]$, $f_1 g_2 - f_2 g_1 \neq 0$, and $f(x_0, y_0) = g(x_0, y_0) = 0$. Then the point $[x_0, y_0]$ is an isolated stationary point of the system $x' = f(x, y)$, $y' = g(x, y)$. Likewise if the point $[0, 0]$ is an isolated stationary point of the linear autonomous system $x' = f_1 x + f_2 y$, $y' = g_1 x + g_2 y$, and his type is node, focus or saddle point, the type of stationary point $[x_0, y_0]$ of the system $x' = f(x, y)$, $y' = g(x, y)$ is the same. If the point $[0, 0]$ is isolated stationary point of the linear autonomous system $x' = f_1 x + f_2 y$, $y' = g_1 x + g_2 y$, and his type is center, the type of stationary point $[x_0, y_0]$ of the system $x' = f(x, y)$, $y' = g(x, y)$ is rotation point or focus.

Now we investigate several special functions $F(z)$ in the equation $z' = F(z)$:

If $F = f(x) + \mathrm{i}g(y)$, then we obtain the autonomous system $x' = f(x)$, $y' = g(y)$. We can solve it as two differential equations $\frac{\mathrm{d}x}{f(x)} = \mathrm{d}t$, $\frac{\mathrm{d}y}{g(y)} = \mathrm{d}t$, where $f \neq 0$, $g \neq 0$, respectively. We can replace this autonomous system by a differential equation $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x)}{g(y)}$ for $f \neq 0$, or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(y)}{f(x)}$ for $g \neq 0$. Hence we have an equation $\frac{y'}{g(y)} = \frac{1}{f(x)}$, where $f \neq 0$, $g \neq 0$.

If $F = f(y) + \mathrm{i}g(x)$, we obtain the autonomous system $x' = f(y)$, $y' = g(x)$. We solve it like an equation $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(y)}{g(x)}$ for $g \neq 0$, or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{f(y)}$ for $f \neq 0$. Hence we have equation $f(y)y' = g(x)$ or $g(x)x' = f(y)$.

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If $F = af(x, y) + \mathrm{i}bf(x, y)$, where $a, b \in \mathbf{R}$, then we obtain the autonomous system $x' = af(x, y)$, $y' = bf(x, y)$. We obtain $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{a}{b}$ for $b \neq 0$, or $\frac{dy}{dx} = \frac{b}{a}$ for $a \neq 0$. This is an equation $bx' = a$, $ay' = b$, respectively, and its solution is $ay - bx = c$, where $c \in \mathbf{R}$.

Especially, for $a = 0$ we obtain $x' = 0$, hence $x(t) = \text{const.}$, $y' = bf(y, c)$, where $c \in \mathbf{R}$, which gives an equation $\frac{dy}{bf(y,c)} = dt$, whose solutions are parallel to yt-plane in $\Omega \times \mathbf{R}$.

Similarly for $b = 0$ we obtain $y' = 0$, hence $y(t) = \text{const.}$, $x' = af(x, c)$, where $c \in \mathbf{R}$, which gives an equation $\frac{dx}{af(x,c)} = dt$, whose solutions are parallel to xt-plane in $\Omega \times \mathbf{R}$.

The aim of this contribution is to construct the collection of examples of differential equations of orthogonal trajectories depending on parameters. In the following examples we investigate orthogonal trajectories of systems of curves like solutions of convenient differential equations. In many cases this method is very useful. We will formulate several geometrical problems, we will solve them using analytical methods and their solutions will be interpreted geometrically.

Examples

Example 1. *Find orthogonal trajectories of the system of concentric circles.*

Without lost of generality we can put the center of the system of concentric circles into the center of the coordinate system. They have the form

$$x^2 + y^2 = c^2, \quad \text{where } c \in \mathbf{R}^+. \tag{3}$$

The differential equation of circles (3) is

$$x + yy' = 0. \tag{4}$$

If we replace y' by $-\frac{1}{y'}$ in (4), we obtain the differential equation of orthogonal trajectories

$$xy' - y = 0. \tag{5}$$

The solution of (5) is the system of lines

$$c_1 y - c_2 x = 0, \quad \text{where } c_1, c_2 \in \mathbf{R}, \tag{6}$$

without the stationary point $[0, 0]$.

The orthogonal trajectories of the system of concentric circles is the system of lines going out from their common center.

Example 2. *Find orthogonal trajectories of the system of circles going through one point and having here the common tangent.*

Without lost of generality we can put the common point of the system of circles into the center of the coordinate system and let x-axis be their common tangent, their centers lie on y-axis. Their equation is

$$x^2 + (y - c)^2 = c^2, \quad \text{where } c \in \mathbf{R}. \tag{7}$$

The differential equation of curves (7) is

$$2xy = \left(x^2 - y^2\right) y'. \tag{8}$$

If we replace y' by $-\frac{1}{y'}$ in (8), we obtain the differential equation of orthogonal trajectories

$$2xyy' = y^2 - x^2. \tag{9}$$

The solution of (9) is a system of circles

$$(x - c)^2 + y^2 = c^2, \quad \text{where } c \in \mathbf{R}. \tag{10}$$

They have centers on x-axis, their common point is the center of the coordinate system and their common tangent is y-axis.

Orthogonal trajectories of the system of circles going through one point and having here the common tangent is the system of circles going through the same point and the tangent of which is perpendicular.

Note that we can rotate the systems of circles (7) and (10) by an angle φ. We can put $\mathrm{tg}\varphi = s$, $-\infty < s < +\infty$, $s \neq 0$. Hence we can these orthogonal systems of circles express as

$$(x - cs)^2 + (y - c)^2 = c^2\left(1 + s^2\right), \quad (x - c)^2 + (y + cs)^2 = c^2\left(1 + s^2\right), \tag{11}$$

where c is any real parameter. They have centers in $[cs, c]$, $[c, -cs]$, respectively, and their radii are $r = c\sqrt{1+s^2}$. They go through the center of the coordinate system and their centers lies on lines having an angle φ with respect to line coordinates. Their differential equations are

$$y'\left(2sxy + y^2 - x^2\right) = sy^2 - sx^2 - 2xy,\ y'\left(2xy - sy^2 + sx^2\right) = y^2 - x^2 + 2sxy. \tag{12}$$

The curves create an orthogonal grid in the plane.

If we translate the center of the system of orthogonal circles (7) and (10) to the point $[a, b]$, $a, b \in \mathbf{R}$, in the plane, we obtain the curves

$$(x-a-c)^2 + (y-b)^2 = c^2, \quad (x-a)^2 + (y-b-c)^2 = c^2, \tag{13}$$

where a and b are fixed constant and $c \in \mathbf{R}$ is parameter. Their differential equations are

$$2(x-a)(y-b)y' = (y-b)^2 - (x-a)^2,\ \left[(x-a)^2 - (y-b)^2\right]y' = 2(x-a)(y-b). \tag{14}$$

Example 3. *Find orthogonal trajectories of the system of equilateral hyperbolas.*

The half-axes of equilateral hyperbolas have the same length and their asymptotes are perpendicular. Without lost of generality the center of hyperbolas can be placed to the center of the coordinate system and their asymptotes can be x-axis and y-axis. Their equation is

$$xy = c, \quad \text{where } c \in \mathbf{R}, \tag{15}$$

and their differential equation is

$$y + xy' = 0. \tag{16}$$

If we replace y' by $-\frac{1}{y'}$ in (16), we obtain the differential equation of orthogonal trajectories

$$yy' = x. \tag{17}$$

The solution of the equation (17) is the system of hyperbolas

$$x^2 - y^2 = c, \quad \text{where } c \in \mathbf{R}. \tag{18}$$

The orthogonal trajectories of the system of equilateral hyperbolas is the system of equilateral hyperbolas, the asymptotes of which are perpendicular.

Example 4. *Find orthogonal trajectories of the parabolas $y^2 = 4c^2(x + c^2)$, where $c \in \boldsymbol{R}$.*

The vertex of the parabolas

$$y^2 = 4c^2\left(x + c^2\right), \quad c \in \mathbf{R}, \tag{19}$$

is in the point $[-c^2, 0]$ and their parameter is $p = 2c^2$. The differential equation of parabolas (19) is

$$y(y')^2 + 2xy' = y. \tag{20}$$

If we replace y' by $-\frac{1}{y'}$ in (20), we obtain identical differential equation. The equation (20) is the second order equation with respect to y'. We can express it like product of two factors

$$\left(y' - \frac{\sqrt{x^2+y^2} - x}{y}\right) \cdot \left(y' + \frac{\sqrt{x^2+y^2} + x}{y}\right) = 0. \tag{21}$$

The first factor in (21) is a differential equation of first the system of curves, the second factor is a differential equation of the second system of curves. Both systems of curves create an orthogonal grid in the plane. If we solve (21), we obtain

$$y^2 = 4k^2 + 4kx, \quad k \in \mathbf{R}. \tag{22}$$

For $k > 0$ the system (22) is the original system of parabolas (19), for $k < 0$ the system (22) is the system of orthogonal parabolas

$$y^2 = -4c^2 \left(x - c^2\right), \quad c \in \mathbf{R}. \tag{23}$$

The parabolas $y^2 = 4c^2 \left(x + c^2\right)$ and $y^2 = -4c^2 \left(x - c^2\right)$, $c \in \mathbf{R}$, create an orthogonal grid in the plain.

In this manner we can create differential equations of orthogonal trajectories:
If we have the differential equation of a system of curves $y' = f(x, y)$, then the differential equation of orthogonal trajectories is $y' = -\frac{1}{f(x,y)}$ and

$$[y' - f(x, y)] \cdot \left[y' + \frac{1}{f(x, y)}\right] = 0 \tag{24}$$

is the differential equation of the orthogonal grid created by two systems of orthogonal curves.

Example 5. *Find orthogonal trajectories of the system of confocal ellipses.*

The confocal ellipses has common foci. Without lost of generality we can put them to the points ± 1 in x-axis. Hence we can express the confocal ellipses as

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - 1} = 1, \quad \text{where } c \in \mathbf{R},\ c > 1. \tag{25}$$

The foci of ellipses (25) are in the points $F[-1, 0]$ and $G[1, 0]$.

We can convert (25) to

$$\frac{x^2}{c^2} - \frac{y^2}{1 - c^2} = 1. \tag{26}$$

For $0 < c < 1$ the equation (26) is equation of confocal hyperbolas. They have common foci at the points $F[-1, 0]$ and $G[1, 0]$.

The differential equation of the system of confocal ellipses is

$$xy\left(y' - \frac{1}{y'}\right) + x^2 - y^2 = 1, \quad y' \neq 0. \tag{27}$$

If we replace y' by $-\frac{1}{y'}$ in (27), we obtain an identical differential equation. The equation (27) is a second order equation with respect to y'. By solving (27) we get two systems of orthogonal curves in the plane – confocal ellipses and confocal hyperbolas. We can express both systems of curves using one equation

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - 1} = 1,$$

where for hyperbolas (26) there is $0 < c < 1$, and for ellipses (25) there is $c > 1$. Both systems of curves have one differential equation (27).

The confocal ellipses and confocal hyperbolas with common foci create an orthogonal grid in the plane.

Example 6. *Find orthogonal trajectories of the system of circles going through two different points in the plain.*

The system of circles going through two different point in the plane create the so called hyperbolical pencil of circles. Both common points are basic points of the pencil. Without lost of generality we can put the points $[0, \pm 1]$ like basic points of the pencil. Hence we can express the hyperbolical pencil of circles

$$(x-c)^2 + y^2 = c^2 + 1, \tag{28}$$

where $c \in \mathbf{R}$. The differential equation of the hyperbolical pencil of circles is

$$2xyy' = y^2 - x^2 - 1. \tag{29}$$

If we replace y' by $-\frac{1}{y'}$ in (29), we obtain the differential equation of orthogonal trajectories

$$\left(1 + x^2 - y^2\right) y' = 2xy. \tag{30}$$

The solution of the equation (30) is

$$x^2 + (y-c)^2 = c^2 - 1, \tag{31}$$

where $c \in \mathbf{R}$, $|c| > 1$. This is equation of the elliptical pencil of circles. Limite points of this pencil are points $[0, \pm 1]$. We can note that the ultimate points are polarly conjugated to all circles of elliptical pencil.

The systems of the elliptical pencil of circles and the hyperbolical pencil of circles create an orthogonal grid in the plane.

Example 7. *Find orthogonal trajectories of the system of parabolas with the common vertex and having here the common tangent.*

Put the common vertex of parabolas without lost of generality to the center of the coordinate system and let their common tangent be an x-axis. They are given by

$$y = cx^2, \quad \text{where } c \in \mathbf{R}. \tag{32}$$

The differential equation of parabolas (32) is

$$xy' = 2y. \tag{33}$$

If we replace y' by $-\frac{1}{y'}$ in (33), we obtain the differential equation of orthogonal trajectories

$$-x = 2yy'. \tag{34}$$

The solution of (34) is a system of ellipses

$$x^2 + 2y^2 + 2c = 0, \quad \text{where } c \in \mathbf{R}. \tag{35}$$

The system of parabolas with the common vertex and which have here the common tangent create an orthogonal grid in the plane with the system of ellipses having constant ratio of length of half-axises, the center of which is in the vertex of parabolas.

We can note that if we find an orthogonal trajectories of the system of curves

$$y + cx^n = 0,$$

where $c \in \mathbf{R}$ and $n \neq 0$ is a fixed integer, we obtain

$$x^2 + ny^2 + k = 0,$$

where $k \in \mathbf{R}$. This is an equation of ellipses for $n > 0$ and $k \in \mathbf{R}^-$, or of a hyperbolas for $n < 0$ and $k \in \mathbf{R}$.

Thus we can generate examples of orthogonal trajectories depending on parameter n.

Example 8. *Find orthogonal trajectories of the system of curves $\mathrm{e}^x \sin y = c$, where $c \in \boldsymbol{R}$.*

The differential equation of the curves $\mathrm{e}^x \sin y = c$, where $c \in \mathbf{R}$, is

$$\mathrm{e}^x \sin y + \mathrm{e}^x \cos y y' = 0.$$

If we replace y' by $-\frac{1}{y'}$ here, we obtain the differential equation of orthogonal trajectories

$$\mathrm{e}^x \left(y' \sin y - \cos y\right) = 0.$$

Their solution is a system of curves $\mathrm{e}^x \cos y = k$, where $k \in \mathbf{R}$.

The systems of curves $\mathrm{e}^x \sin y = c$ and $\mathrm{e}^x \cos y = c$, where $c \in \mathbf{R}$, create an orthogonal grid in the plane.

Example 9. *Find orthogonal trajectories of the system of parabolas $y^2 = a\,(x - c)$, where $c \in \boldsymbol{R}$ and a is a fixed real constant.*

The differential equation of the system of curves $y^2 = a\,(x - c)$, where $c \in \mathbf{R}$ is arbitrary and a is a fixed real constant, $a \neq 0$, is

$$2yy' = a. \tag{36}$$

If we replace y' by $-\frac{1}{y'}$ in (36), we obtain the differential equation of orthogonal trajectories

$$-2y = ay'. \tag{37}$$

The solution of (37) is

$$2x + a \ln |y| = c, \quad \text{where} \;\; c \in \mathbf{R},\; a \in \mathbf{R},\; a \neq 0 \text{ fixed.} \tag{38}$$

We can express these curves like exponential functions $y = ce^{-\frac{2x}{a}}$, where $c \in \mathbf{R}$ and a is a fixed real constant, $a \neq 0$.

The systems of parabolas $y^2 = a\,(x - c)$ and of exponential functions $y = ce^{-\frac{2x}{a}}$, where $c \in \mathbf{R}$ and a is fixed real constant, $a \neq 0$, create an orthogonal grid in the plane.

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