# A NOTE ON THE TOPOLOGY GENERATED BY SCOTT OPEN FILTERS

Martin Maria Kovár

*(University of Technology, Brno)*

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Abstract. In Problem 527 of Open Problems in Topology J. Lawson and M. Mislove ask, for which DCPO's the Scott topology has a basis of open filters and for which DCPO's the topology generated by the Scott open filters is $T_0$. In this paper we introduce a notion of a Hofmann-Mislove DCPO, for which we can partially answer the question.

## 0. Introduction and Terminology

In this section we explain the most of used terminology, with an exception of some primitives and notions that we touch only marginally. These notions are not essential for understanding the paper. However, for more complex and detailed explanation and introductory to the topics the reader is referred to the books and monographs [G$^+$], [LM] and [Vi].

Let $P$ be a set, $\leq$ a reflexive and transitive, but not necessarily antisymmetric binary relation on $P$. Then we say that $\leq$ is a *preorder* on $P$ and $(P\ \leq)$ is a *preordered set*. For any subset $A$ of a preordered set $(P, \leq)$ we denote $\uparrow A = \{x|\ x \geqslant y \text{ for some } y \in A\}$ and $\downarrow A = \{x|\ x \leqslant y \text{ for some } y \in A\}$. An important example of a preordered set is given by a *preorder of specialization* of a topological space $(X, \tau)$, which is defined by $x \leq y$ if and only if $x \in \mathrm{cl}\,\{y\}$. This preorder is a partial order in the usual sense if and only if the space $(X, \tau)$ is $T_0$. For any $x \in X$ it is obvious that $\downarrow\{x\} = \mathrm{cl}\,\{x\}$. A set is said to be *saturated* in $(X, \tau)$ if it is an intersection of open sets. One can easily verify that a set $A \subseteq X$ is saturated in $(X, \tau)$ if and only if $A = \uparrow A$, that is, if and only if $A$ is an upper set with respect to the preorder of specialization of $(X, \tau)$. Thus for every set $B \subseteq X$, the set $\uparrow B$ we call a *saturation* of $B$. The family of all compact saturated sets in $(X, \tau)$ is a closed

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base for a topology $\tau^d$, which is called *de Groot dual* of the original topology $\tau$. A topological space is said to be *sober* if it is $\mathrm{T}_0$ and every irreducible closed set is a closure of a singleton.

Let $(X, \leq)$ be a *partially ordered set*, or briefly, a *poset*. If $(X, \leq)$ has, in addition, finite meets, then any element $p \in X$ is said to be *prime* if $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$ for every $x, y \in X$. The set $P$ of all prime elements of $(X, \leq)$ is called the *spectrum* of $(X, \leq)$. If $(X, \leq)$ is a frame and $P \subseteq X$ is its spectrum, by a *locale corresponding to* $(X, \leq)$ we mean the quadruple $(X, P, \leq, \vDash)$, where $\vDash$ is a binary relation defined by $p \vDash x$ if and only if $x \nleq p$. Then the elements of $X$ we call *opens* and the elements of $P$ the *abstract* or *localic* points of the locale $(X, P, \leq, \vDash)$. A locale is called *spatial* if it can arise from the frame of open sets of some topological space. Then $X$ can be represented as a topology on $P$, $\leq$ is the set inclusion and the meaning of the relation $\vDash$ is the same as of $\in$. In that case, the topological space $(P, X)$ is sober. It is not difficult to show that a locale is spatial if and only if it has enough abstract points to distinguish the opens, i.e. $u = v$ if and only if $(p \vDash u) \Leftrightarrow (p \vDash v)$ for every $p \in P$ and $u, v \in X$. For more detail regarding the other possible definitions of a locale, its properties and relationship to the topological spaces we refer the reader to [Vi].

We say that the poset $(X, \leq)$ is *directed complete*, or *DCPO*, if every directed subset of $X$ has a least upper bound – a supremum. A subset $U \subseteq X$ is said to be *Scott open*, if $U = \uparrow U$ and whenever $D \subseteq X$ is a directed set with $\sup D \in U$, then $U \cap D \neq \varnothing$. One can easily check that the Scott open sets of a DCPO form a topology. This topology we call the *Scott topology*. Thus a set $A \subseteq X$ is closed in the Scott topology if and only if $A = \downarrow A$ and if $D \subseteq A$ is directed, then $\sup D \in A$. It follows from Zorn's Lemma that in a DCPO, every element of a Scott closed subset is comparable with some maximal element. It is easy to see that the closure of a singleton $\{x\}$ in the Scott topology is $\downarrow\{x\}$, thus the original order $\leq$ of $X$ can be recovered from the Scott topology as the preorder of specialization. In particular, the Scott topology on a DCPO is always $\mathrm{T}_0$.

Let us describe some other important topologies on DCPO's. The *upper topology* [$\mathrm{G}^+$], which is also referred as the *weak topology* [LM] or the *lower interval topology* [La] has the collection of all principal lower sets $\downarrow\{x\}$, where $x \in X$, as the subbase for closed sets. The preorder of specialization of the lower interval topology coincide with the original order of $(X \leq)$. Hence, the saturation of a subset $A \subseteq X$ with respect to this topology is $\uparrow A$. Similarly, the *lower topology*, also referred as the *weak$^d$ topology* [LM] or the *upper interval topology* [La], arises from its subbase for closed sets which consists of all principal upper sets $\uparrow\{x\}$, where $x \in X$. Note that the weak$^d$ topology is not the de Groot dual of the weak topology in general; the weak$^d$ topology is the weak topology with respect to the inverse partial order. The preorder of specialization of the upper interval topology is a binary relation inverse to the original order of $(X, \leq)$. Consequently, the saturation of a subset $A \subseteq X$ with respect to this topology is $\downarrow A$. The topology on the spectrum $P$ of a directed complete $\wedge$-semilattice $(X, \leq)$, induced by the upper interval topology, is called the *hull-kernel topology* [LM].

The set $F \subseteq X$ is called a filter, if $F = \uparrow F$ and every finite subset of $F$ has a lower bound in $(X, \leq)$. An interesting topology on $X$ can arise from the family of all

filters in $(X, \leq)$, which are open in the Scott topology. We can take this family as a subbase for open sets of the new topology. This topology is strictly weaker than the Scott topology in general. However, J. Lawson and M. Mislove stated the question when these two topologies coincide as a part of the Problem 527 in Open Problems in Topology [LM]. The complete question was stated as follows:

**Problem 527.** *Characterize those DCPO's*

(i) *for which the Scott topology has a basis of open filters, and*
(ii) *for which the topology generated by Scott open filters is $T_0$.*

*Analogously, characterize those $T_0$ topological spaces $X$ for which the Scott topology on the lattice $O(X)$ of open sets satisfies (i) or (ii).*

Note that both conditions are clearly true in the first case if the DCPO is continuous and, consequently, in the second case if $X$ is core compact.

## 1. Preliminaries

We start with study of the relationship between compact sets and Scott open filters in DCPO's and we also introduce the notions of a Hofmann-Mislove DCPO and its generalized spectrum. A source of inspiration of this section, among others, was an interesting paper [KP] of K. Keimel and J. Paseka. In the following definition we slightly adjust the notion of a prime element to be more applicable also for those DCPO's which do not have finite meets.

**Definition 1.1.** Let $(X, \leq)$ be a poset, $L \subseteq X$. We say that $L$ is *prime* if for every $a, b \in X$

$$\downarrow\{a\} \cap \downarrow\{b\} \subseteq \downarrow L \Rightarrow (a \in \downarrow L) \vee (b \in \downarrow L).$$

It can be easily seen that if $(X, \leq)$ has finite meets, any element $p \in X$ is prime if and only if the singleton $\{p\}$ is prime as a set. Hence, we can extend the notion of a prime element also to those posets which do not necessarily have finite meets. Thus in the following text we mean that an element $p$ of a poset $(X, \leq)$ is prime if and only if $\{p\}$ is prime in the sense of the previous definition. As the following lemma shows, the notions of a prime set and of a filter are dual.

**Lemma 1.2.** *Let $(X, \leq)$ be a poset. Then $L \subseteq X$ is prime if and only if $F = X \setminus \downarrow L$ is a filter.*

*Proof.* Let $L \subseteq X$ be prime. Then $F = X \setminus \downarrow L$ is a lower set. Let $a, b \in F$. Then $a, b \notin \downarrow L$, which implies that there is some $c \in \downarrow\{a\} \cap \downarrow\{b\}$ such that $c \notin \downarrow L$, i.e. $c \in F$. Conversely, let $F$ be a filter. Suppose that $\downarrow\{a\} \cap \downarrow\{b\} \subseteq \downarrow L$ for some $a, b \in X$. Then $a, b \notin \downarrow L$ implies $a, b \in F$ which means that there is some $c \leq a$, $c \leq b$, $c \in F$. Then $c \in \downarrow\{a\} \cap \downarrow\{b\}$, but $c \notin L$, which is a contradiction. $\square$

**Definition 1.3.** Let $X$ be a set, $\Phi \subseteq 2^X$. We say that $K \subseteq X$ is *up-filtered compact with respect to the family* $\Phi$ if $K \cap (\bigcap \varphi) \neq \varnothing$ for every filter base $\varphi \subseteq \Phi$ such that every its element meets $K$. In a DCPO $(X, \leq)$ we say that $K \subseteq X$ is *up-filtered compact*, if it is up-filtered compact with respect to the family $\{\uparrow\{x\} \mid x \in X\}$ of principal upper sets.

If the DCPO $(X, \leq)$ has finite meets, up-filtered compact means the same as compact with respect to the upper interval topology. Note that this clone of compactness we refer as *up-compact* in [Ko]. The following lemma shows that there is some "de Groot-like" duality between the upper interval topology and the Scott topology.

**Lemma 1.4.** *Let $(X, \leq)$ be a DCPO. Then $K \subseteq X$ is Scott closed if and only if $K$ is saturated in the upper interval topology and up-filtered compact.*

*Proof.* Let $K \subseteq X$ be Scott closed. Then $F = X \smallsetminus K$ is an upper set and so $F = \bigcup_{a \in F} \uparrow\{a\}$ which means that $K = X \smallsetminus \bigcup_{a \in F} \uparrow\{a\} = \bigcap_{a \in F}(X \smallsetminus \uparrow\{a\})$. Then $K$ is an intersection of open sets in the upper interval topology, so it is saturated. Let $\varphi = \{\uparrow\{a\} \,|\, a \in A\}$ be a filter base such that $K \cap \uparrow\{a\} \neq \varnothing$ for every $a \in A$. Since $\varphi$ is a filter base then if $a, b \in A$ there exists $c \in A$ such that $\uparrow\{c\} \subseteq \uparrow\{a\} \cap \{b\}$, that is, $c \geq a, b$. So $A$ is directed. Further, if $a \in A$, then there is some $x \in K \cap \uparrow\{a\}$. It follows that $a \leq x$ and since $K$ is a lower set, we have $a \in K$. Hence $A \subseteq K$. Then $u = \sup A \in K$ since $K$ is Scott closed. But then $u \in K \cap (\bigcap_{a \in A} \uparrow\{a\})$. It means that $K$ is up-filtered compact.

Conversely, let $K \subseteq X$ be up-filtered compact and saturated in the upper interval topology. Then there exists $F \subseteq X$ such that $K = \bigcap_{a \in F}(X \smallsetminus \uparrow\{a\})$ and, consequently, $K$ is a lower set. Let $A \subseteq K$ be directed. Then $\Phi = \{\uparrow\{a\} \,|\, a \in A\}$ is a closed filter base and every its element clearly meets $K$. Since $K$ is up-filtered compact, there exists $t \in K \cap (\bigcap_{a \in A} \uparrow\{a\})$. Then $t \geq a$ for every $a \in A$, so $t \geq \sup A$. Since $K$ is a lower set, $\sup A \in K$. Hence, $K$ is Scott closed. □

**Proposition 1.5.** *Let $(X, \leq)$ be a DCPO, $\omega$ the upper interval topology on $X$, $\omega_P$ the induced topology on $P \subseteq X$. The following conditions (i) and (ii) are equivalent:*

(i) *There exist $P \subseteq X$ such that:*
   *(1) For every Scott open filter $F \subseteq X$, if we denote $\psi(x) = P \smallsetminus \uparrow\{x\}$ and $L = \bigcap_{a \in F} \psi(a)$, the set $L$ is up-filtered compact, saturated in $(P, \omega_P)$ and $F = \{x | \, x \in X, L \subseteq \psi(x)\}$.*
   *(2) For every up-filtered compact and saturated $L \subseteq P$ in $(P, \omega_P)$, the set $F = \{x | \, x \in X, L \subseteq \psi(x)\}$ is a Scott open filter.*

(ii) *There exist $P \subseteq X$ such that:*
   *(1) For every Scott closed prime set $K \subseteq X$, the set $L = P \cap K$ is up-filtered compact, saturated in $(P, \omega_P)$ and $K = \downarrow L$.*
   *(2) For every up-filtered compact and saturated $L \subseteq P$ in $(P, \omega_P)$, the set $\downarrow L$ is Scott closed and prime.*

*Proof.* It is obvious that $F \subseteq X$ is a Scott open filter if and only if $K = X \smallsetminus F$ is a Scott closed prime set. Further, $\bigcap_{a \in F} \psi(a) = \bigcap_{a \in F}(P \smallsetminus \uparrow\{x\}) = P \smallsetminus \bigcup_{a \in F} \uparrow\{a\} = P \smallsetminus F = P \cap K$ and $X \smallsetminus \downarrow L = \{x | \, x \in X, x \notin \downarrow L\} = \{x | \, x \in X, L \cap \uparrow\{x\} = \varnothing\} = \{x | \, x \in X, L \subseteq \psi(x)\}$. Now it is clear that (ii) is only a reformulation of (i). □

**Definition 1.6.** Let $(X, \leq)$ be a DCPO. We say that $(X, \leq)$ is *Hofmann-Mislove*, if $(X, \leq)$ satisfies any of the conditions (i) or (ii) of Proposition 1.5. The set $P \subseteq X$ from (i) or (ii) we call a *generalized spectrum* of $(X, \leq)$; the topology $\omega_P$ we call the *generalized hull-kernel topology* on $P$.

The next proposition shows that in a Hofmann-Mislove DCPO, the generalized spectrum is determined uniquely.

**Proposition 1.7.** *Let $(X, \leq)$ be a Hofmann-Mislove DCPO, $S \subseteq X$ its any generalized spectrum. Then $S = \{p | \, p \in X, p$ is prime$\} = \{m | \, m$ is a maximal element of a Scott closed prime subset of $X\}$.*

*Proof.* Let $M = \{m | \, m$ is a maximal element of a Scott closed prime set$\}$, $P = \{p | \, p \in X, p$ is prime$\}$. Let $p \in P$ be a prime element. Then $\downarrow\{p\}$ is a Scott closed prime set and $p$ its maximal element. Then $P \subseteq M$. Let $m \in M$ and let $K \subseteq X$ be a Scott closed prime set such that $m \in K$ is its maximal element. Then $K = \downarrow(K \cap S)$, so $m \in \downarrow(K \cap S)$. Then there exists $t \in K \cap S$ with $m \leq t$. But $m$ is maximal in $K$, so $m = t \in S$. Hence, $M \subseteq S$. Let $s \in S$. Then $\{s\}$ is clearly up-filtered compact, as well as $\downarrow\{s\}$. The set $L = S \cap \downarrow\{s\}$ is saturated in $(S, \omega_S)$ and $\downarrow L = \downarrow\{s\}$. Hence, $L$ is up-filtered compact. Then $\downarrow L$ is Scott closed and prime, which means, in particular, that $s$ is prime. Therefore, $S \subseteq P$. Now we have $P \subseteq M \subseteq S \subseteq P$, which completes the proof. $\square$

*Remark 1.8.* If $(X, \leq)$ is a DCPO with finite meets, that is, a directed complete $\wedge$-semilattice, then the natural requirement (2) of the conditions (i) or (ii) in Proposition 1.5 is fulfilled automatically. Indeed, from Proposition 1.7 we know that $P$ must be the set of all prime elements. Let $L \subseteq P$ be any set and let $\downarrow\{a\} \cap \downarrow\{b\} \subseteq \downarrow L$ for some $a, b \in X$. Then $a \wedge b \in \downarrow L$, so there is some $p \in L$ such that $a \wedge b \leq p$. But then $a \leq p$ or $b \leq p$ since $p$ is prime. But then $a \in \downarrow L$ or $b \in \downarrow L$. Hence, $L$ and also $\downarrow L$ are prime sets. If $L$ is, in addition, up-filtered compact, then so $\downarrow L$ and by Lemma 1.4, $\downarrow L$ is Scott closed. $\square$

Hence, from the condition (i) of Proposition 1.5 we can see that the Hofmann-Mislove DCPO's are – in other words – exactly those DCPO's which satisfy the well-known Hofmann-Mislove Theorem.

**Corollary 1.9.** *Let $(X, \leq)$ be a DCPO with finite meets. Then $(X, \leq)$ is Hofmann-Mislove if and only if every maximal element of a Scott closed prime set is prime.*

*Proof.* The "only if" part follows from Proposition 1.7. Let $K \subseteq X$ be a Scott closed prime set, $P$ the set of all prime elements. Let $L = P \cap K$. It follows from Lemma 1.4 that $K$ is up-filtered compact and saturated in the upper interval topology, hence $L$ is saturated in the hull-kernel topology on $P$. We will show that $K = \downarrow L$. Obviously, $\downarrow L \subseteq K$ since $K$ is a lower set. But every $x \in K$ is comparable with some maximal element $m \in K$, which is prime by the assumption. Hence $x \leq m \in L$, so $m \in \downarrow L$. It holds $K = \downarrow L$, but then $L$ is also up-filtered compact. Then the part (1) of the condition (ii) in Proposition 1.5 is fulfilled and the part (2) is fulfilled automatically. This completes the proof. $\square$

The following corollary is a reformulation of the previously mentioned Hofmann-Mislove Theorem. In addition to the assumptions of Corollary 1.9 we will suppose that the DCPO $(X, \leq)$ is a distributive lattice. But this is equivalent to the assumption that $(X, \leq)$ is a frame.

**Corollary 1.10.** *Let $(X, \leq)$ be a frame. Then $(X, \leq)$ is Hofmann-Mislove.*

*Proof.* Let $K \subseteq X$ be a Scott closed prime set, $m \in K$ its maximal element. Suppose that $a \wedge b \leq m$ for some $a, b \in X$. Then $(a \vee m) \wedge (b \vee m) = (a \wedge b) \vee (a \wedge m) \vee (m \wedge b) \vee (m \wedge m) = m$. Then $\downarrow\{a \vee m\} \cap \downarrow\{b \vee m\} \subseteq K$, but $K$ is lower and prime, so $a \vee m \in K$ or $b \vee m \in K$. But then $a \vee m = m$ or $b \vee m = m$, i.e. $a \leq m$ or $b \leq m$, since $m$ is maximal. Hence, $m$ is a prime element. $\square$

## 2. Scott open sets and filters

Now we give a partial answer to the Problem 527 in terms of Hofmann-Mislove DCPO's, their generalized spectra and the de Groot-like duality between the upper interval topology and the Scott open topology. If not otherwise specified, thorough this section all saturations of sets as well as up-filtered compactness are assumed with respect to the upper interval topology on a DCPO.

**Theorem 2.1.** *Let $(X, \leq)$ be a Hofmann-Mislove DCPO, $P \subseteq X$ its generalized spectrum. The following conditions are equivalent:*

(i) *The Scott topology on $X$ has a base of Scott open filters.*
(ii) *For every up-filtered compact saturated set $K \subseteq X$ there is a family $\{K_i\}_{i \in I}$ of up-filtered compact sets such that $K = \bigcap_{i \in I} K_i$ and $K_i = \downarrow(P \cap K_i)$.*
(iii) *Every up-filtered compact saturated set in $X$ is an intersection of saturations of up-filtered compact subsets of the generalized spectrum of $X$.*

*Proof.* The conditions (ii) and (iii) are clearly equivalent since $L \subseteq P$ is up-filtered compact if and only if $\downarrow L$ is up-filtered compact. Suppose (i). Let $K \subseteq$ be a up-filtered compact set, saturated in the upper interval topology. Then, by Lemma 1.4, $F = X \smallsetminus K$ is a Scott open set. By (i), $F = \bigcup_{i \in I} F_i$, where every $F_i$ is a Scott open filter. Denote $K_i = X \smallsetminus F_i$. Then $K = \bigcap_{i \in I} K_i$ and by Lemma 1.2 every $K_i$ is a Scott closed prime set. Since $(X \leq)$ is Hofmann-Mislove, by the definition $L_i = P \cap K_i$ is up-filtered compact subset of the generalized spectrum and $K_i = \downarrow L_i$, which gives (iii).

Suppose (ii). Let $F \subseteq X$ be a Scott open set. We put $K = X \smallsetminus F$. It follows from Lemma 1.4 that $K$ is up-filtered compact and saturated in the upper interval topology. By (ii), $K = \bigcap_{i \in I} K_i$, where every $K_i$ is up-filtered compact as well as $L_i = P \cap K_i$ and $K_i = \downarrow L_i$. Since $(X \leq)$ is Hofmann-Mislove, by the definition $K_i$ is Scott closed and prime, so by Lemma 1.2 $F_i = X \smallsetminus K_i$ is a Scott open filter and clearly, $F = \bigcup_{i \in I} F_i$, which implies (i). $\square$

**Corollary 2.2.** *Let $(X, \leq)$ be a frame. Then the Scott topology on $X$ is the de Groot dual $\omega^d$ of the upper interval topology $\omega$. Moreover, the Scott topology on $X$ has a base of Scott open filters if and only if the saturations of compacts subsets of the spectrum of $X$ form a closed base of $\omega^d$.*

Now we characterize those DCPO's, whose topology generated by Scott open filters is $\mathrm{T}_0$.

**Theorem 2.3.** *Let $(X, \leq)$ be a Hofmann-Mislove DCPO, $P \subseteq X$ its generalized spectrum. The following conditions are equivalent:*

(i) *The topology on $X$ generated by the Scott open filters is $T_0$.*
(ii) *For every $x, y \in X$ there exists up-filtered compact $L \subseteq P$ such that one of the points $x, y$ is and the other is not contained in the saturation of $L$.*
(iii) *For every $x, y \in X$, $x \neq y$ there is a prime element $p \in X$ such that $x \leq p$, $y \nleq p$ or $x \nleq p$, $y \leq p$ .*

*Proof.* Let $x, y \in X$. Then, by Lemma 1.2 there is a Scott open filter $F \subseteq X$ such that $x \in F$, $y \notin F$ if and only if there is a Scott closed prime set $K \subseteq X$ such that $x \notin K$, $y \in K$. But in a Hofmann-Mislove DCPO, every Scott closed prime set is a saturation of a up-filtered compact subset of the generalized spectrum, and, conversely, a saturation of a up-filtered compact subset of the generalized spectrum is Scott closed and prime. Thus (i) and (ii) are equivalent. The implication (iii) $\Rightarrow$ (ii) is clear. It remains to show that (ii) $\Rightarrow$ (iii).

Suppose(ii). Let $x, y \in X$ and let $L \subseteq P$ be up-filtered compact, such that $x \in \downarrow L$ and $y \notin \downarrow L$. Then $\downarrow L$ is a Scott closed prime set, so $\downarrow L$ has some maximal element, say $p$, such that $x \leq p$. Clearly, $y \nleq p$ since otherwise $y \in \downarrow L$. But by Proposition 1.7 $p$ is prime. Hence, (iii) holds. $\square$

For frames we can say even more. The $\mathrm{T}_0$ separation property of the topology generated by Scott open filters on a frame $(X, \leq)$ is equivalent to the spatiality of the corresponding locale, as it follows from the next two corollaries.

**Corollary 2.4.** *Let $(X, \tau)$ be a topological space, $(\tau, \subseteq)$ the frame of open sets. Then the topology on $\tau$ generated by the Scott open filters is $T_0$.*

*Proof.* Let $U, V \in \tau$ and $U \neq V$. Then $U \nsubseteq V$ or $U \nsupseteq V$. Suppose the first possibility (the other one is analogous). Then there is some $p \in U$ such that $p \notin V$. Then $p \in X \smallsetminus V$, which implies $\mathrm{cl}\,\{p\} \subseteq X \smallsetminus V$ and so $V \cap \mathrm{cl}\,\{p\} = \varnothing$. Denote $P = X \smallsetminus \mathrm{cl}\,\{p\}$. Then $V \subseteq P$ and $U \nsubseteq P$. We will show that $P \in \tau$ is prime. Let $R, S \in \tau$ such that $R \cap S \subseteq P$. Then $p \notin R \cap S$, so $p \notin R$ or $p \notin S$. Since $R$, $S$ are open, it follows that $R \cap \mathrm{cl}\,\{p\} = \varnothing$ or $S \cap \mathrm{cl}\,\{p\} = \varnothing$. But then $R \subseteq P$ or $S \subseteq P$. Hence, $P$ is prime. The Theorem 2.3 now completes the proof. $\square$

**Corollary 2.5.** *Let $(X, \leq)$ be a frame and let the topology on $X$ generated by Scott open filters be $T_0$. Then the corresponding locale is spatial.*

*Proof.* Let $P \subseteq X$ be the spectrum of $(X, \leq)$. Then the abstract points of the corresponding locale $(X, P, \leq, \vDash)$ can be represented as the elements of $P$. For every $p \in P$ and $u \in X$ we have

$$p \vDash u \Leftrightarrow u \nleq p \Leftrightarrow p \in X \smallsetminus \uparrow\{u\}\,.$$

It follows from Theorem 2.3 that any two different opens $x, y \in X$ of the locale $(X, P, \leq, \vDash)$ have an abstract point $p \in P$ such that either $p \vDash x$ and $p \nvDash y$, or $p \nvDash x$ and $p \vDash y$. Hence $(X, P, \leq, \vDash)$ is spatial. $\square$

Since the Scott topology on a DCPO is always $T_0$, we have the following corollary.

**Corollary 2.6.** *Let $(X, \leq)$ be a frame whose Scott topology has a base of open filters. Then the corresponding locale is spatial, or equivalently, $(X, \leq)$ can be represented as a topology of some sober space.*

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Department of Mathematics, Faculty of electrical engineering and communication, University of Technology, Technická 8, Brno, 616 69, Czech Republic

*E-mail address*: kovar@feec.vutbr.cz

# THE DE GROOT DUAL FOR TOPOLOGICAL SYSTEMS AND LOCALES

Martin Maria Kovár

*(University of Technology, Brno)*

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Abstract. A topology is the de Groot dual of another topology, if it has a closed base consisting of all compact saturated sets. In this paper we extend the de Groot dual also to topological systems and locales. We compare the behavior of this extension with the properties of the de Groot dual of a topological space, and study the similarities as well as the differences. We state a natural open question: For which locales is the extended de Groot dual operator of finite order?

## 0. Introduction

J. D. Lawson and M. Mislove stated a question whether a sequence of iterated duals of a topology terminates by two topologies, which are dual to each other (1990, Problem 540 of Open Problems in Topology,[LM]). It should be noted that for $T_1$ spaces, the problem was solved a long time before it was formulated by Lawson and Mislove, by G. E. Strecker, J. de Groot and E. Wattel (1966, [GSW]). In $T_1$ spaces, the dual studied by Lawson and Mislove coincides with another dual, studied by de Groot, Strecker and Wattel more than 30 years ago. That is why the dual studied by Lawson and Mislove is now often called the de Groot dual, although de Groot originally introduced and studied the dualization for $T_1$ spaces only. This natural unification of the terminology is due to R. Kopperman [Kop] who probably came with it first. In 2000, it was B. Burdick who proved that for some topologies on hyperspaces (which are not $T_1$ in general) there can arise at most four distinct topologies from iterating the dualization process: the original topology $\tau$, then $\tau^d$, $\tau^{dd}$ and $\tau^{ddd}$, since for these topologies held $\tau^{dd} = \tau^{dddd}$. Finally, this result was generalized for all topological spaces by the author (see [Kov]) in the spring of 2001.

Although the original Problem 540 of Lawson and Mislove arose from certain questions in domain theory, the connections between this part of general topology and one of its main sources of motivation, the theoretical computer science, need not be obvious to everyone. Perhaps that is (among other reasons) also because the language of theoretical computer science is more adjusted to use the terms of order, lattices, frames

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or domains, and if it uses some topological principles, rather than classical topological notions it is related to topological systems and locales, in which the thinking need not be *spatial*. In this paper we try to bring the de Groot dual from topological spaces to its implicit origins in theoretical computer science and study what happens with the dual if we leave the realm of spatial thinking.

## 1. Dualizations of Topological Systems

Before we start, let us recall some notions which will be essential for our further constructions. Let $(X, \tau)$ be a topological space. By $\tau^d$ we denote a topology *dual* to $\tau$, that is, the topology generated by the compact saturated sets in $(X, \tau)$, as its closed base. This topology is also often called *co-compact* and the operator $d$ we call *de Groot dual operator*, as it has been explained in the introduction. Let $A$ be a frame, $X$ a set, $\models \subseteq X \times A$. We write $x \models a$ for $(x, a) \in \models$ and say "x satisfies a". Let the following conditions are satisfied:

(i) If $B \subseteq A$, then $(x \models \bigvee B) \Leftrightarrow (x \models b$ for some $b \in B)$.
(ii) If $C \subseteq A$ is finite, then $(x \models \bigwedge C) \Leftrightarrow (x \models c$ for every $c \in C)$.

Then we say that the triple $(X, A, \models)$ forms a *topological system*. The elements of $A$ we call *opens* [Vi]. Obviously, if $(X, \tau)$ is a topological space, then $(X, \tau, \in)$ is a topological system. Conversely, let $(X, A, \models)$ be a topological system. Let $\mathrm{int}_X(a) = \{x | \, x \in X, x \models a\}$ for every $a \in A$ and let $\tau_X(A) = \{\mathrm{int}_X(a) | \, a \in A\}$. One can easily check that $\tau_X(A)$ is a topology on $X$ and so $(X, \tau_X(A))$ is a topological space. We call $\tau_X(A)$ the *underlying topology* on $X$ induced by $A$. The corresponding topological system $(X, \tau_X(A), \in)$ is said to be the *spatialization* of $(X, A, \models)$. The notions of compactness and saturation can be naturally and without any difficulties extended from topological spaces to topological systems. A set $K \subset X$ is said to be *compact (saturated, respectively)* in $(X, A, \models)$ if $K$ is compact (saturated, respectively) in the topological space $(X, \tau_X(A))$. Let us denote $\uparrow K$ the intersection of all open sets of $(X, \tau_X(A))$, containing $K$. We call $\uparrow K$ the *saturation* of $K$ in $(X, A, \models)$. Obviously, $\uparrow K$ is saturated in $(X, \tau_X(A))$. A set $U \subset X$ is said to be *co-compact (co-saturated, respectively)* in $(X, A, \models)$ if $X \smallsetminus U$ is compact (saturated, respectively) in $(X, A, \models)$. Let $(X, A, \vdash)$ and $(Y, B, \models)$ be topological systems. Let $f : X \to Y$ a mapping, $g : B \to A$ be a frame morphism such that $x \vdash g(b)$ if and only if $f(x) \models b$ for every $x \in X, b \in B$. Then the couple $(f, g)$ is called *a morphism* or *a continuous map* from $(X, A, \vdash)$ to $(Y, B, \models)$. We denote by $\mathbf{2} = \{\bot, \top\}$ the *Serpiński frame*, consisting of the two elements $\top$ – the top and $\bot$ – the bottom.

However, the way how the dualization of a topological system can be defined consistently with the dualization of a topological space, need not be unique. Independently on its definition, it seems to be reasonable if the dualization again is a topological system and if it commutes with the spatialization if possible. To cover the most general case, we say that the topological system $(X', P, \models)$ is a *dualization* of $(X, A, \vdash)$ *represented* by the frame $P$ if there exists a morphism $(f, e)$ of topological systems from $(X, (\tau_X(A))^d, \in)$ to $(X', P, \models)$ such that $e : P \to (\tau_X(A))^d$ is a frame epimorphism. The dualization $(X', P, \models)$ of $(X, A, \vdash)$ is *strict* if $f$ is a bijection. The basic properties of the defined concepts are described by the following theorem.

**Theorem 1.1.** *Let $(X, A, \vdash)$ be a topological system, $P$ be a frame. The following conditions are equivalent:*

(i) *$P$ represents a dualization of $(X, A, \vdash)$.*
(ii) *$P$ represents a strict dualization of $(X, A, \vdash)$.*

(iii) *There exists a frame epimorphism* $e : P \to (\tau_X(A))^d$.

*Proof.* The implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) are trivial. Suppose (iii). For every $x \in X$, $p \in P$ we define $x \models p$ if and only if $x \in e(p)$. Suppose that $x \in X$ and $S \subseteq P$. Then $x \models \bigvee S \Leftrightarrow x \in e(\bigvee S) = \bigcup_{s\in S} e(s) \Leftrightarrow (\exists s \in S) : (x \in e(s)) \Leftrightarrow (\exists s \in S) : (x \models s)$. Let $F \subseteq P$ be finite. Then $x \models \bigwedge F \Leftrightarrow x \in e(\bigwedge F) = \bigcap_{s\in F} e(s) \Leftrightarrow (\forall s \in F) : (x \in e(s)) \Leftrightarrow (\forall s \in F) : (x \models s)$. Hence, $(X, P, \models)$ is a topological system which is a strict dualization of $(X, A, \vdash)$. $\square$

Now we show how the dualization commutes with the spatialization. Let $(X', P, \models)$ be a dualization of $(X, A, \vdash)$. Then there exist $f : X \to X'$ and a frame epimorphism $e : P \to (\tau_X(A))^d$ such that $(f, e)$ is a morphism from $(X, ((\tau_X(A))^d, \in)$ to $(X', P, \models)$. The spatialization of $(X, A, \vdash)$ is $(X, \tau_X(A), \in)$, whose dualization (in topological spaces) is $(X, (\tau_X(A))^d, \in)$. The dualization of $(X, A, \vdash)$ is $(X', P, \models)$, whose spatialization is $(X', \tau_{X'}(P), \in)$. Let $p \in P$. Then $f^{-1}(\mathrm{int}_{X'}(p)) = e(p) \in (\tau_X(A))^d$. Hence, $f$ is a continuous mapping of topological spaces $(X, (\tau_X(A))^d)$ and $(X', \tau_{X'}(P))$. Moreover, if $U \in (\tau_X(A))^d$, then there exists $p \in P$ such that $e(p) = U$. Then $f(U) = \mathrm{int}_{X'}(p) \cap f(X)$, so $f : X \to f(X)$ is an open mapping. Hence, the dualization $(X', P, \models)$ is strict if and only if $f$ is a homeomorphism. This can be illustrated by the following diagram, in which $d$ denotes the de Groot dual operator, "Spat" is the spatialization and "Dual" stands for the studied dualization:

$$\begin{tikzcd}
(X, A, \vdash) \arrow[rrr, "\text{Dual}"] \arrow[d, "\text{Spat}"'] & & & (X', P, \models) \arrow[d, "\text{Spat}"] \\
(X, \tau_X(A), \in) \arrow[r, "d"'] & (X, (\tau_X(A))^d, \in) \arrow[r, "f"'] \arrow[urr, "(f,e)"] & & (X', \tau_{X'}(P), \in)
\end{tikzcd}$$

**Corollary 1.2.** *Suppose that $P$ is a frame that represents a (strict) dualization of the topological system $(X, A, \vdash)$. If there is a frame $S$ and a frame epimorphism $f : S \to P$, then also $S$ represents a (strict) dualization of $(X, A, \vdash)$.*

By the definition, for every dualization $(X', P, \models)$ of $(X, A, \vdash)$ there exists a morphism $(f, e)$ from $(X, (\tau_X(A))^d, \in)$ to $(X', P, \models)$ which, however, need not be unique. In this sense, $(X, (\tau_X(A))^d, \in)$ can be interpreted as the least or (weakly) initial dualization of $(X, A, \vdash)$, which we will consider as the extension of the de Groot dual to topological systems. In other words, the *de Groot dual for topological systems* is the composition

$$d \circ \mathrm{Spat}\,.$$

Let $\mathrm{Loc}(X', P, \models)$ be the localification of $(X', P, \models)$. Then $\mathrm{Loc}(X', P, \models)$ is again a dualization of $(X, A, \vdash)$. Let $(X', P, \models)$ be a locale. Then $(f, e)$ can be extended over $\mathrm{Loc}(X, (\tau_X(A))^d, \in)$, so $\mathrm{Loc}(X, (\tau_X(A))^d, \in)$ can be interpreted as the least or (weakly) initial localic dualization of $(X, A, \vdash)$. This dualization we will take as the analogue of the de Groot dual for locales. In other words, the *de Groot dual for locales* we will define as the composition

$$l = \mathrm{Loc} \circ d \circ \mathrm{Spat},$$

where "Loc" means the localification. On the other hand, it can be easily seen that the "greatest" or "terminal" counterpart in dualizations of topological systems does not exist. The result of the previous considerations we can formulate as a corollary.

**Corollary 1.3.** *Every topological system has its least dualization and its least localic dualization.*

The next following corollaries summarize some other facts about dualizations of topological systems.

**Corollary 1.4.** *Let $(X_i, A_i, \models_i)$ be a topological system and $P_i$ a frame representing its (strict) dualization for every $i \in I$. Then the product frame $P = \prod_{i\in I} P_i$ represents a (strict) dualization of $(X_i, A_i, \models_i)$ for every $i \in I$.*

We say that the topological system $(X_n, A_n, \models_n)$ is the $n$-th iterated (strict) dualization of $(X, A, \vdash)$ if there exist topological systems $(X_i, A_i, \models_i)$ for $i = 1, 2, \dots, n-1$ such that $(X_1, A_1, \models_1)$ is a (strict) dualization of $(X, A, \vdash)$ and $(X_{i+1}, A_{i+1}, \models_{i+1})$ is a (strict) dualization of $(X_i, A_i, \models_i)$ for $i = 1, 2, \dots, n-1$.

**Corollary 1.5.** *Let $(X, A, \models)$ be a topological system. Then there exists a frame $P$ representing its $n$-th iterated (strict) dualization for every $n \in \mathbb{N}$.*

The last corollary follows from the fact that the strict dualization commutes with the spatialization and the fact that $\tau^{dd} = \tau^{dddd}$ holds for every topological space $(X, \tau)$ [Kov].

**Corollary 1.6.** *A frame $P$ represents the $n$-th iterated strict dualization of a topological system $(X, A, \vdash)$ if and only if $P$ represents its $(n+2k)$-th iterated strict dualization, for every $n = 2, 3, \dots$ and $k = 1, 2, \dots$.*

It is a natural question whether every dualization of a locale is again a localic topological system. However, it is easy to answer this question in the negative. To give a proper counterexample, we need to recall some notions. A subset $A$ of a topological space is *irreducible* if it is not a union of two proper closed subsets. A topological space is said to be *sober* if it is $\mathrm{T}_0$ and every irreducible closed set is a closure of a singleton. It is well-known that for spatial topological systems (i.e. topological spaces), sober and localic mean the same.

**Example 1.7.** (i) Let $X = \mathbb{N}$, $A = \{0, 1, \dots, \omega\}$ with its natural linear order ($\omega$ is the first infinite ordinal) and we put $x \vdash a \Leftrightarrow x \leq a$ for every $x \in X$ and $a \in A$. Then $(X, A, \vdash)$ is a spatial topological system. It follows that $\tau = \tau_X(A) = \{\varnothing, X\} \cup \{\{1, 2, \dots, n\} \mid n \in \mathbb{N}\}$ is $\mathrm{T}_0$ and the specialization order is inverse to the natural order of $X = \mathbb{N}$. Every non-empty closed subset of $(X, \tau)$ is of the form $G_k = \{k, k+1, \dots\}$, where $k \in \mathbb{N}$, it is irreducible and $G_k = \mathrm{cl}\,\{k\}$. So the topological space $(X, \tau)$ is sober and $(X, A, \vdash)$ is a locale.

(ii) The compact saturated subsets of $(X, \tau)$ are exactly the finite lower closed sets. Then $\tau^d = \{\varnothing\} \cup \{\{n, n+1, \dots, \} \mid n \in \mathbb{N}\}$ an hence, every non-empty closed set is either finite or equal to $X$. Hence, $X$ is irreducible, but there is no point $x \in X$ with $X = \mathrm{cl}^d\,\{x\}$. Hence, $(X, \tau^d)$ is not sober and so $(X, \tau^d, \in)$ is not a locale.

(iii) The soberification of $(X, \tau^d)$ can be interpreted as the topological space $(Y, \sigma)$, where $Y = \{1, 2, \dots, \omega\}$ and $\sigma = \{\varnothing\} \cup \{\{n, n+1, \dots, \omega\} \mid n \in \mathbb{N}\}$. Indeed, $f : \tau^d \to \mathbf{2}$ is a frame morphism if and only if it is monotone. All the morphisms are represented by the points of $X$ with the exception of the morphism $f_\omega$ for which $f_\omega(U) = \top$ if and only if $U \neq \varnothing$. This morphism is represented in $Y$ by the new point $\omega$ which satisfies all non-empty opens. Every non-empty closed set in $(Y, \sigma)$ is irreducible and has the form $F_k = \{1, 2, \dots, k\} = \mathrm{cl}_\sigma\,\{k\}$ or equals to $Y = \mathrm{cl}_\sigma\,\{\omega\}$.

(iv) The specialization order of $(Y, \sigma)$ coincides with its natural linear order and the compact saturated sets in $(Y, \sigma)$ are exactly all upper-closed subsets of $Y$. Hence,

$\sigma^d = \{\varnothing, \mathbb{N}, Y\} \cup \{\{1, 2, \ldots, n\} \,|\, n \in \mathbb{N}\}$. The non-empty closed sets in $(Y, \sigma^d)$ are irreducible and have the form $\{\omega\} = \mathrm{cl}_{\sigma^d}\{\omega\}$ and $H_k = \{k, k+1, \ldots, \omega\} = \mathrm{cl}_{\sigma^d}\{k\}$ for every $k \in \mathbb{N}$. Since $\sigma^d$ is $\mathrm{T}_0$, $(Y, \sigma^d)$ is sober.

(v) The specialization order of $(Y, \sigma^d)$ is inverse to its natural linear order. Then every compact saturated subset of $(Y, \sigma^d)$ equals to $Y$ if it contains $\omega$ and is finite lower closed otherwise. Then $\sigma^{dd} = \{\varnothing\} \cup \{\{n, n+1, \ldots, \omega\} \,|\, n \in \mathbb{N}\} = \sigma$. Hence, in terms of the de Groot dual for locales, $\tau^l = \sigma = \sigma^{ll} = \tau^{lll}$. $\square$

According to Example 1.7 (v), the following problem, analogous to Problem 540 of Lawson and Mislove that we mentioned in the introduction, has its natural place here.

**Problem 1.8.** *Characterize those locales (topological spaces, respectively) for which the process of iterating the dual operator $l$ can generate at most finitely many non-isomorphic frames (up to frame isomorphism distinct topologies, respectively).*

## 2. Dualizations represented by frames of functions

The way through the de Groot dual composed with the spatialization and the localification which we described in the previous section is not very straightforward. An alternative way how we can dualize topological systems and locales will be studied in this section. But firstly, let us mention a one of possible motivations with a background in the computer science.

Let $X$ be a set of some computer programs and $A$ be a poset of their possible initial outputs $a \in A$. We write $x \vdash a$ if the program $x$ has the output $a$. The set $K \subseteq X$ of programs is *compact* in some sense if the following condition is satisfied: If the programs in $K$ has their initial outputs in some directed $B \subseteq A$, then there is some $b \in B$ such that every program from $K$ has the output $b$. The set $K \subseteq X$ is saturated if it is the largest set with the same initial outputs. The program $x$ may be considered *independent* on a compact set of programs $K$ if the event that some program in $K$ has an output $a$ does not imply the event that also $x$ has the output $a$. From the observations whether a program $x$ is independent on a compact saturated set $K$ may arise a new structure which may be interpreted as the dual (in some sense) of the original triple $(X, A, \vdash)$.

Now we formalize the construction that we have sketched above. Let $\mathbf{2}^A$ be the set of all mappings of $A$ to $\mathbf{2}$, where we put $y \leq z$ if and only if $y(a) \leq z(a)$ for every $a \in A$. Obviously, $\mathbf{2}^A$ is a frame isomorphic to the power set $2^A$ ordered by inclusion. Let $P \subseteq \mathbf{2}^A$. By **False** and **True** we denote the constant functions on $A$ identically equal to $\bot$ and $\top$. Let us denote $[P] = \bigcap \big\{S \,|\, P \subseteq S \subseteq \mathbf{2}^A,\ S$ is closed under all joins and finite meets in $\mathbf{2}^A\big\}$.

**Lemma 2.1.** *Let $A$ be a frame, $P \subseteq \mathbf{2}^A$. The following statements are fulfilled:*

(i) *$[P]$ is a subframe of $\mathbf{2}^A$.*
(ii) *If $P$ is closed under finite meets, then every element of $[P]$ is a join of some $Y \subseteq P$.*
(iii) *If $P$ is closed under finite meets and every its element distinct from* **True** *preserves (directed, non-empty, non-empty and directed, respectively) joins, then every element of $[P]$ distinct from* **True** *preserves (directed, non-empty, non-empty and directed, respectively) joins.*
(iv) *If $P$ is closed under finite meets and finite joins, then every element of $[P]$ is a join of some directed $Y \subseteq P$.*

*Proof.* To show (i), we will prove that $[P]$ is closed under all joins and finite meets. Let $\Phi = \big\{F \,|\, P \subseteq F \subseteq \mathbf{2}^A,\ F$ is closed under all joins and finite meets in $\mathbf{2}^A\big\}$. Let

$Y \subseteq [P]$ and let $Z \subseteq [P]$ be finite. Let $F \in \Phi$. Then $Y, Z \subseteq F$ and hence $\bigvee Y \in F$ and $\bigwedge Z \in F$, which implies that $\bigvee Y, \bigwedge Z \in \bigcap_{F \in \Phi} F = [P]$.

Let us prove (ii). If $P$ is closed under finite meets, then $\mathbf{True} = \bigwedge \varnothing \in P$. Let $F = \{y | \, y \in \mathbf{2}^A, y = \bigvee Y \text{ for some } Y \subseteq P\}$. Then $\mathbf{False} = \bigvee \varnothing \in F$ and $\mathbf{True} = \bigvee \{\mathbf{True}\} \in F$. Let $Y \subseteq F$ be non-empty. For every $y \in Y$ there exists $Y_y \subseteq P$ such that $y = \bigvee Y_y$. Denote $w = \bigvee Y$ and $W = \bigcup_{y \in Y} Y_y$ and let $v \in W$. There is some $y \in Y$ such that $v \in Y_y$. Then $v \leq y \leq w$, so $w$ is an upper bound of $W$. Let $u \in \mathbf{2}^A$ be an upper bound of $W$. Then $u$ is an upper bound of $Y_y$ for every $y \in Y$. It follows that $u \geq y$ for every $y \in Y$, so $u \geq w$. It follows that $w = \bigvee W$. Since $W \subseteq P$, we have $w \in P$. Let $s, z \in F$. There are some $Y_s, Y_z \subseteq P$ such that $s = \bigvee Y_s$ and $z = \bigvee Y_z$. Denote $V = \{p \wedge q | \, p \in Y_s, q \in Y_z\}$. Since $P$ has binary meets, it follows that $V \subseteq P$. We will show that $s \wedge z = \bigvee V$. Obviously, $s$, $z$ are upper bounds of $V$, so $s \wedge z$ is an upper bound of $V$. Let $u \in \mathbf{2}^A$ be an upper bound of $V$. Then $s \wedge z = (\bigvee Y_s) \wedge z = \bigvee_{p \in Y_s} (p \wedge z) = \bigvee_{p \in Y_s} \bigvee_{q \in} (p \wedge q) \leq u$. It follows that $s \wedge z = \bigvee V$ and hence $s \wedge z \in F$. It follows that $F$ is closed under all joins and binary meets, so $P \subseteq [P] \subseteq F$. Hence, every element of $[P]$ is a join of some $Y \subseteq P$.

Now, let us show (iii). Let $y \in [P]$, $y \neq \mathbf{True}$. Since $P$ has finite meets, by (ii) there exists some $Y \subseteq P$ such that $y = \bigvee Y$. Then $z \neq \mathbf{True}$ for every $z \in Y$. Let $B$ be a (directed, non-empty, non-empty and directed, respectively) subset of $A$. If $B = \varnothing$ and the elements of $P$ distinct from $\mathbf{True}$ preserve the empty join, then $y(\bigvee \varnothing) = y(\bot) = (\bigvee Y)(\bot) = \bigvee_{z \in Y} z(\bot) = \bigvee_{z \in Y} \bot = \bot$. If $B \neq \varnothing$, it follows that $y(\bigvee B) = (\bigvee Y)(\bigvee B) = \bigvee_{z \in Y} z(\bigvee B) = \bigvee_{z \in Y} \bigvee_{b \in B} z(b) = \bigvee_{b \in B} \bigvee_{z \in Y} z(b) = \bigvee_{b \in B} (\bigvee_{z \in Y} z)(b) = \bigvee_{b \in B} (\bigvee Y)(b) = \bigvee_{b \in B} y(b)$.

Finally, let us prove (iv). If $P$ has finite meets and joins, by (ii) it follows that there is some $Y \subseteq P$ such that $y = \bigvee Y$. We put $V = \{\bigvee F | \, \varnothing \neq F \subseteq Y, \text{F is finite}\}$. It follows that $V$ is directed, $V \subseteq P$ and $y = \bigvee V$. $\square$

Now, let $(X, A, \models)$ be a topological system, $a \in A$ and $M \subseteq X$. We denote

$$w_M(a) = \begin{cases} \top, & \text{if } x \models a \text{ for every } x \in M, \\ \bot, & \text{otherwise.} \end{cases}$$

**Lemma 2.2.** *Let $(X, A, \models)$ be a topological system and $K$ be compact in $(X, A, \models)$. If $K \neq \varnothing$ then $w_K$ preserves directed joins and finite meets; $w_\varnothing = \mathbf{True}$.*

*Proof.* It is obvious that $w_\varnothing = \mathbf{True}$. Let $K \neq \varnothing$. Then $w_K(\top) = \top$ and $w_K(\bot) = \bot$. Let $B \subseteq A$ be non-empty and directed. Suppose that $w_K(\bigvee B) = \top$. Then $x \models \bigvee B$ for every $x \in K$. Since $(X, A, \models)$ is a topological system, it follows that for every $x \in K$ there is some $b_x \in B$ such that $x \models b_x$. It follows that $\{\mathrm{int}_X(b_x) | \, x \in K\}$ is an open cover of $K$. Since $K$ is compact, there exist $x_1, x_2, \ldots, x_k \in K$ such that $K \subseteq \bigcup_{i=1}^k \mathrm{int}_X(b_{x_i})$. Let $b_K \in B$ be such element that $b_K \geq b_{x_i}$ for every $i = 1, 2, \ldots, k$. Then $K \subseteq \mathrm{int}_X(b_K)$. It follows that $w_K(b_K) = \top$ and hence $\bigvee_{b \in B} w_K(b) = \top$. Conversely, suppose that $\bigvee_{b \in B} w_K(b) = \top$. Then, there is some $b_K \in B$ such that $w_K(b_K) = \top$. If follows that $K \subseteq \mathrm{int}_X(b_K)$. Since $b_K \leq \bigvee B$ it follows that $K \subseteq \mathrm{int}_X(\bigvee B)$, which gives $w_K(\bigvee B) = \top$.

Let $C \subseteq A$ be non-empty and finite. Then $w_K(\bigwedge C) = \top$ if and only if $x \models \bigwedge C$ for every $x \in K$. This is true if and only if $x \models c$ for every $x \in K$ and $c \in C$. But this is equivalent to $w_K(c) = \top$ for every $c \in C$. This holds if and only if $\bigwedge_{c \in C} w_K(c) = \top$. $\square$

Let $y : A \to \mathbf{2}$ be a mapping and $(X, A, \models)$ a topological system. We say that $y$ is a *set (compact, finite, point,* respectively) *function* in $(X, A, \models)$ if there exists a set (a compact set, a finite set, a singleton, respectively) $K \subseteq X$ such that $y = w_K$. If $K = \{p\}$, we write $w_K = w_{\{p\}} = w_p$. The set containing all compact functions in $(X, A, \models)$ we denote by $\mathcal{C}(X, A)$. For every mapping $y : A \to \mathbf{2}$ we put

$$c(y) = \begin{cases} \{x \mid x \in X, (\forall a \in A) : (y(a) = \top) \Rightarrow (x \models a)\}, & \text{if } y \neq \mathbf{True}, \\ \varnothing, & \text{if } y = \mathbf{True}. \end{cases}$$

**Lemma 2.3.** *Let $(X, A, \models)$ be a topological system and $y \in \mathbf{2}^A$. Then the set $c(y)$ is saturated in $(X, A, \models)$.*

*Proof.* Obviously, $\varnothing$ is saturated. Let $c(y) \neq \varnothing$. Suppose that $z \in X \smallsetminus c(y)$. Then there is some $a_0 \in A$ such that $y(a_0) = \top$ and $z \not\models a_0$. Then for every $x \in c(y)$ it follows that $x \models a_0$, which implies that $c(y) \subseteq \mathrm{int}_X(a_0)$ and $z \notin \mathrm{int}_X(a_0)$. Therefore, $c(y) = \bigcap \{\mathrm{int}_X(a) \mid a \in A, c(y) \subseteq \mathrm{int}_X(a)\}$. $\square$

**Lemma 2.4.** *Let $(X, A, \models)$ be a topological system and $K \subseteq X$ be compact in $(X, A, \models)$. Then $w_K = w_{\uparrow K}$ and $c(w_K) = \uparrow K$.*

*Proof.* If $a \in A$, then $w_K(a) = \top \Leftrightarrow K \subseteq \mathrm{int}_X(a) \Leftrightarrow \uparrow K \subseteq \mathrm{int}_X(a) \Leftrightarrow w_{\uparrow K}(a) = \top$. Hence $w_K = w_{\uparrow K}$. Let $x \in c(w_K)$. It follows that if $K \subseteq \mathrm{int}_X(a)$ for some $a \in A$, then $x \in \mathrm{int}_X(a)$. Then $x \in \bigcap \{\mathrm{int}_X(a) \mid K \subseteq \mathrm{int}_X(a)\} = \uparrow K$. Conversely, let $x \in \uparrow K$. If $w_K(a) = \top$ for some $a \in A$, then $K \subseteq \mathrm{int}_X(a)$, which implies that $x \in \mathrm{int}_X(a)$ and so $x \models a$. Hence $x \in c(w_K)$. $\square$

**Lemma 2.5.** *Let $(X, A, \models)$ be a topological system. The set $\mathcal{C}(X, A)$ is closed under finite meets in $\mathbf{2}^A$.*

*Proof.* From Lemma 2.2 it follows that $\bigwedge \varnothing = \mathbf{True} = w_\varnothing \in \mathcal{C}(X, A)$. Let $y, z \in \mathcal{C}(X, A)$. Denote $K = c(y) \cup c(z)$ and $w = w_K$. It follows that $w(a) = \top \Leftrightarrow K = c(y) \cup c(z) \subseteq \mathrm{int}_X(a) \Leftrightarrow (y(a) = \top) \wedge (z(a) = \top) \Leftrightarrow (y \wedge z)(a) = \top$. It follows that $w = y \wedge z$, which implies that $y \wedge z \in \mathcal{C}(X, A)$. $\square$

We denote by $\Phi^d(X, A)$ the set of all compact saturated sets in a topological system $(X, A, \vdash)$ and by $\Sigma^d(X, A)$ the set of all co-compact co-saturated sets in $(X, A, \vdash)$. Obviously, $\Phi^d(X, A)$ is a union semilattice and $\Sigma^d(X, A)$ is an intersection semilattice, isomorphic to $\Phi^d(X, A)$.

**Corollary 2.6.** *Let $(X, A, \models)$ be a topological system. Then $c : \mathcal{C}(X, A) \to \Phi^d(X, A)$ is a semilattice isomorphism.*

*Proof.* In the proof of Lemma 2.5 we proved that $c(y \wedge z) = c(y) \cup c(z)$ for $y, z \in \mathcal{C}(X, A)$. From Lemma 2.4 it follows that $c$ is a one-to-one. $\square$

Let $(X, A, \vdash)$ be a topological system. We will say that $x \in X$ is *independent* on $y \in [\mathcal{C}(X, A)]$ and write $x \models y$ if there is some $a \in A$ such that $y(a) = \top$ and $x \nvdash a$.

**Theorem 2.7.** *Let $(X, A, \vdash)$ be a topological system. Then $(X, [\mathcal{C}(X, A)], \models)$ is a topological system which is a strict dualization of $(X, A, \vdash)$.*

*Proof.* Firstly, will prove that $(X, [\mathcal{C}(X, A)], \models)$ forms a topological system. Since $\mathbf{False}(a) = \bot$ for every $a \in A$, it follows $x \models \mathbf{False} = \bigvee \varnothing$ for no $x \in X$. Let $x \in X$.

Since $x \nvdash \bot$ and $\mathbf{True}(\bot) = \top$ it follows $x \models \mathbf{True} = \bigwedge \varnothing$. Let $Y \subseteq [\mathcal{C}(X,A)]$ be non-empty. It follows that $x \models \bigvee Y$ if and only if there exists $a \in A$ such that $x \nvdash a$ and $(\bigvee Y)(a) = \bigvee_{y \in Y} y(a) = \top$. But this is true if and only if there is some $a \in A$ and $y \in Y$, such that $x \nvdash a$ and $y(a) = \top$. This is equivalent to existence of some $y \in Y$ such that $x \models y$. Let $Z \subseteq [\mathcal{C}(X,A)]$ be non-empty and finite. It follows that $x \models \bigwedge Z$ if and only if there exists $a \in A$ such that $x \nvdash a$ and $(\bigwedge Z)(a) = \bigwedge_{z \in Z} z(a) = \top$. But this holds if and only if there is some $a \in A$ such that $x \nvdash a$ and $z(a) = \top$ for every $z \in Z$. This implies that $x \models z$ for every $z \in Z$. Conversely, let $x \models z$ for every $z \in Z$. Then, for every $z \in Z$, there exists $a_z \in A$ such that $z(a_z) = \top$ and $x \nvdash a_z$. We put $a = \bigvee_{z \in Z} a_z$. Then $z(a) = \top$ and $x \nvdash a$ for every $z \in Z$, which is equivalent to $x \models \bigwedge Z$. Therefore, $(X, [\mathcal{C}(X,A)], \models)$ is a topological system.

Now, we will show that $[\mathcal{C}(X,A)]$ induces the co-compact topology on $X$. We put $s(U) = w_{X \smallsetminus U}$ for every $U \in \Sigma^d(X,A)$. Let $U \in \Sigma^d(X,A)$ and $x \in \mathrm{int}_X(s(U))$. Then $x \models w_{(X \smallsetminus U)}$, which implies that there exists $a \in A$ such that $w_{(X \smallsetminus U)}(a) = \top$ and $x \nvdash a$. It follows that $X \smallsetminus U \subseteq \mathrm{int}_X(a)$ and hence $x \notin X \smallsetminus U$. Hence $x \in U$. Conversely, let $x \in U$. Then $x \notin X \smallsetminus U = K$, where $X \smallsetminus U$ is compact and saturated. Hence, there exists some $a \in A$ such that $x \notin \mathrm{int}_X(a)$ and $X \smallsetminus U \subseteq \mathrm{int}_X(a)$. Then $s(U)(a) = w_{(X \smallsetminus U)}(a) = \top$ and $x \nvdash a$, which implies that $x \models s(U)$ and so $x \in \mathrm{int}_X(s(U))$. It follows that $U = \mathrm{int}_X(s(U))$ for every $U \in \Sigma^d(X,A)$. But $\Sigma^d(X,A)$ forms a base of the dual topology $(\tau_X(A))^d$, and hence $(\tau_X(A))^d \subseteq \tau_X([\mathcal{C}(X,A)])$. Let $y \in [\mathcal{C}(X,A)]$ and $x \in \mathrm{int}_X(y)$. Then $x \models y$. It follows from Lemma 2.1 (ii) that there is some $Y \subseteq \mathcal{C}(X,A)$ such that $y = \bigvee Y$. Then $x \models y_x$ for some $y_x \in Y$ and so $x \in \mathrm{int}_X(y_x)$. It follows that $y_x$ is a compact function, so there exist $K$ compact and saturated in $(X, A, \vdash)$ such that $y_x = w_K$. Let $U = X \smallsetminus K$. Then $U \in (\tau_X(A))^d$ and $s(U) = y_x$. If follows that $x \in U = \mathrm{int}_X(s(U)) = \mathrm{int}_X(y_x) \subseteq \mathrm{int}_X(y)$. Hence, $\mathrm{int}_X(y)$ is a union of elements of $(\tau_X(A))^d$, and so $(\tau_X(A))^d = \tau_X([\mathcal{C}(X,A)])$. Now, the pair $(f,e) = (\mathrm{id}_X, \mathrm{Spat})$ is the requested morphism of topological systems from the definition of the strict dualization. $\square$

Let $A$ be a poset. We denote by $\langle A \to \mathbf{2} \rangle \subseteq \mathbf{2}^A$ the set of all functions $A \to \mathbf{2}$ that preserve all directed joins and finite meets, whenever they exist. We also denote $\langle A \to \mathbf{2} \rangle^T = \langle A \to \mathbf{2} \rangle \cup \{\mathbf{True}\}$. We write $[A \to \mathbf{2}]$ instead of $[\langle A \to \mathbf{2} \rangle]$.

**Proposition 2.8.** *Let $(X, A, \vdash)$ be a topological system. The following statements are equivalent:*

(i) $\mathcal{C}(X,A) = \langle A \to \mathbf{2} \rangle^T$.
(ii) *Every element of $\langle A \to \mathbf{2} \rangle$ is a compact function.*
(iii) *Every element of $\langle A \to \mathbf{2} \rangle$ is a set function.*
(iv) *Every element of $\langle A \to \mathbf{2} \rangle$ is a meet of point functions.*

*Proof.* Obviously, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Suppose (iii). Let $y \in \langle A \to \mathbf{2} \rangle$. There is some $M \subseteq X$ such that $y = w_M$. We will show that $y = \bigwedge_{p \in M} w_p$. Suppose that $y(a) = \top$ for some $a \in A$. Then $M \subseteq \mathrm{int}_X(a)$, which implies that $w_p(a) = \top$ for every $p \in M$. Hence, $(\bigwedge_{p \in M} w_p)(a) = \bigwedge_{p \in M} w_p(a) = \top$. Conversely, let $(\bigwedge_{p \in M} w_p)(a) = \top$. Then $w_p(a) = \top$ and hence $p \in \mathrm{int}_X(a)$ for every $p \in M$. It follows that $M \subseteq \mathrm{int}_X(a)$, which implies that $y(a) = w_M(a) = \top$. It follows that $y = \bigwedge_{p \in M} w_p$, so (iv) is true.

Suppose (iv). Let $y \in \langle A \to \mathbf{2} \rangle$. There is some $M \subseteq X$ such that $y = \bigwedge_{p \in M} w_p = w_M$. We will show that $M$ is compact. Suppose that $B \subset A$ is directed and $M \subseteq \bigcup_{b \in B} \mathrm{int}_X(b) = \mathrm{int}_X(\bigvee B)$. Then $w_M(\bigvee B) = \bigvee_{b \in B} w_M(b) = \top$. It follows that there

is some $b_1 \in B$ such that $w_M(b_1) = \top$. Hence, $M \subseteq \mathrm{int}_X(b_1)$, which implies that $M$ is compact, so $y = w_M \in \mathcal{C}(X, A)$. It follows (i). $\square$

Let $A$ be a frame. Recall that a map $y : A \to \mathbf{2}$ is a *frame morphism* if $y$ preserves all joins and all finite meets.

**Lemma 2.9.** *Let $A$ be a frame. Then every element $y \in \langle A \to \mathbf{2}\rangle$ is a meet of frame morphisms.*

*Proof.* We denote $F = \{u|\, u : A \to \mathbf{2}$ is a frame morphism$, u \geq y\}$ and $z = \bigwedge F$. It follows that $z \geq y$. To show that $z = y$ we will prove that if $y(a) = \bot$ for some $a \in A$, then also $z(a) = \bot$. Let us denote $M = \{b|\, b \in A, a \leq b, y(b) = \bot\}$. Let $L \subseteq M$ be a linearly ordered chain. Then $y(\bigvee L) = \bigvee_{b\in L} y(b) = \bot$, so $\bigvee L \in M$ and $L$ has an upper bound. Hence, by Zorn's Lemma, $M$ has some maximal element $m \in M$. Then $y(m) = \bot$, which implies that $m \neq \top \in A$.Let

$$u(b) = \begin{cases} \bot, & \text{if } b \leq m, \\ \top, & \text{otherwise.} \end{cases}$$

Obviously, $u(\bot) = \bot$ and $u(\top) = \top$. Let $\varnothing \neq B \subseteq A$. Then $u(\bigvee B) = \bot \Leftrightarrow \bigvee B \leq m \Leftrightarrow (\forall b \in B) : (b \leq m) \Leftrightarrow (\forall b \in B) : (u(b) = \bot) \Leftrightarrow \bigvee_{b\in B} u(b) = \bot$. Let $C \subseteq A$ be non-empty and finite. Then $u(\bigwedge C) \leq \bigwedge_{c\in C} u(c)$, since $u$ is monotone. Suppose that $u(\bigwedge C) = \bot$ and $u(c) = \top$ for every $c \in C$. Then, for any $c \in C$, it follows that $c \nleq m$ and hence $m < m \vee c$. Then $\top = \bigwedge_{c\in C} y(m \vee c) = y(\bigwedge_{c\in C}(m \vee c)) = y(m \vee (\bigwedge C))$, which implies that $\bigwedge C \nleq m$. This is a contradiction to $u(\bigwedge C) = \bot$. $\square$

**Theorem 2.10.** *Let $(X, A, \vdash)$ be a locale. Then $(X, [A \to \mathbf{2}], \models)$ is a strict (not necessarily localic) dualization of $(X, A, \vdash)$.*

*Proof.* The points of $X$ are exactly the frame morphisms of $A \to \mathbf{2}$, so they coincide with the point functions. The theorem now follows from Theorem 2.7, Proposition 2.8 and Lemma 2.9. $\square$

Note that the points of the locale $(X, A, \vdash)$ are fully determined by the corresponding frame $A$, so we can identify $(X, A, \vdash)$ with $A$ and the localic dualization of $(X, A, \vdash)$ we can similarly identify with the frame $[A \to \mathbf{2}]$.

**Corollary 2.11.** *Let $(X, A, \vdash)$ be a locale. Then the iterated localic dualizations of $(X, A, \vdash)$ are represented by the sequence of frames $[A \to \mathbf{2}], [[A \to \mathbf{2}] \to \mathbf{2}], [[[A \to \mathbf{2}] \to \mathbf{2}] \to \mathbf{2}], \ldots$.*

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Department of Mathematics, Faculty of electrical engineering and communication, University of Technology, Technická 8, Brno, 616 69, Czech Republic

*E-mail address*: `kovar@feec.vutbr.cz`